Moser inequalities in Gauss space

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We deal with a family of limiting exponential type Sobolev inequalities in Gauss space, namely the space $\mathbb{R}^n$ endowed with the Gauss probability measure γ_n given by

$$\mathrm{d}\gamma_n(x) = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}\,\mathrm{d}x \quad \text{for } x \in \mathbb{R}^n.$$

The inequalities to be considered admit diverse variants. All of them concern the validity, for $\beta > 0$, of the uniform bound

$$\int_{\mathbb{R}^n} e^{(\kappa|u|)^{\frac{2\beta}{2+\beta}}}\,\mathrm{d}\gamma_n \le C \tag{1}$$

for some positive constants κ and C, and for every weakly differentiable function u in $\mathbb{R}^n$ subject to a constraint on a form of exponential integrability for $|\nabla u|^\beta$, and to the normalization that the mean value or the median of u is equals to zero. The most straightforward version of the relevant constraint reads

$$\int_{\mathbb{R}^n} e^{|\nabla u|^\beta}\,\mathrm{d}\gamma_n \le M \tag{2}$$

for some constant $M > 1$. It is known that for every $\beta > 0$, the exponent $\frac{2\beta}{2+\beta}$ is the largest possible that makes the inequality (1) true for suitable κ and C.

Our main focus will be on a sharp form of inequality (1). Specifically, we investigate the optimal (i.e. largest possible) constant κ for which inequality (1) holds under the normalization condition (2) or some alternate closely related assumption.

We will next discuss the question of existence of maximizers for (1) under the relevant constraints. We also comment the main and surprising dissimilarities between this type of inequalities in Gauss and Euclidean space.