The Minkowski problem in Gaussian probability space

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Motivation

The (classical) Minkowski problem

Let μ be a finite Borel measure on S^{n-1}. Find the necessary and sufficient conditions so that μ is the *surface area measure* of a convex body K.
To what extent is the solution unique?

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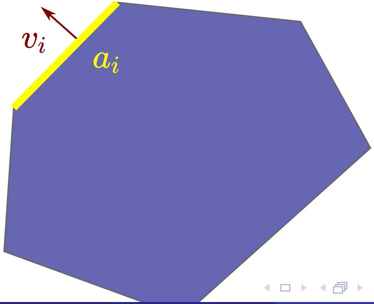
$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega))$$

Given $v_1, \cdots, v_N \in S^{n-1}$
$a_1, \cdots, a_N > 0$



Motivation

Solve

$$\mu(\cdot) = S_K(\cdot).$$

When K is $C^{2,+}$,

$$S_K(v) = \frac{1}{H_{n-1}(K, v)} dv.$$

When $\mu = f dv$, Monge-Ampère equation

$$\det(h_{ij} + h\delta_{ij}) = f.$$

Minkowski, Aleksandrov, Fenchel, Jessen, Cheng, Yau, Pogorelov, Caffarelli,...

Motivation

Why study *surface area measure*?

Aleksandrov's variational formula

$$\frac{d}{dt}\bigg|_{t=0} V(K+tL) = \int_{S^{n-1}} h_L(v)\,dS_K(v).$$

Moral of the story: surface area measure is the "derivative" of volume.

Motivation

The Brunn-Minkowski inequality

$$V((1-t)K+tL)^{\frac{1}{n}} \geq (1-t)V(K)^{\frac{1}{n}} + tV(L)^{\frac{1}{n}},$$

"$=$" iff K and L are homothetic.

The Minkowski inequality

$$\frac{1}{n}V_1(K,L) =: \frac{1}{n}\int_{S^{n-1}} h_L(v)dS_K(v) \geq V(L)^{\frac{1}{n}}V(K)^{\frac{n-1}{n}},$$

"$=$" iff K and L are homothetic.

The BM inequality is closely connected to the Minkowski problem.

Motivation

The Minkowski inequality hints

$$\inf_{h_L}\left\{\frac{1}{n}\int_{S^{n-1}} h_L(v)d\mu : V(L)=1\right\}$$

to solve the Minkowski problem.

- If $\mu = S_K$ and $V(L)=1$, then the MI states:

$$\frac{1}{n}\int_{S^{n-1}} h_L d\mu = \frac{1}{n}\int_{S^{n-1}} h_L dS_K \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}} = V(K)^{\frac{n-1}{n}}$$

with "=" iff K and L are homothetic.

Motivation

The Minkowski inequality implies the uniqueness of the solution.

- If $S_K = S_L$, then

$$V(L) = \frac{1}{n}\int_{S^{n-1}} h_L dS_L = \frac{1}{n}\int_{S^{n-1}} h_L dS_K \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}$$

  Therefore, $V(L) \geq V(K)$.
  Using the symmetry of the above argument, we see equality holds. Hence K and L are translations of each other.

Existence and uniqueness of the Minkowski problem also implies the Minkowski inequality with equality condition. (Klain, 2004)

Solution to the classical Minkowski problem

There exists a solution K to the equation

$$\mu = S_K$$

if and only if μ is not concentrated in any proper subspaces and

$$\int_{S^{n-1}} v d\mu(v) = o.$$

The solution is unique up to a translation.

Minkowski problems—recent development

- The L_p Minkowski problem (Lutwak, 1993 & 1996)

$$h^{1-p}\det(h_{ij}+h\delta_{ij})=f$$

  - $p=0$: log-Minkowski problem, log-Brunn-Minkowski conjecture.
  - $p=-n$: centro-affine Minkowski problem.

- The (L_p) dual Minkowski problem (Huang-LYZ 2016):

$$(h^2+|\nabla h|^2)^{\frac{q-n}{2}}h^{1-p}\det(h_{ij}+h\delta_{ij})=f$$

- The (L_p) chord Minkowski problem (Xi-LYZ 2023):

$$h^{1-p}\widetilde{V}_{q-1}(K,Dh)\det\left(h_{ij}+h\delta_{ij}\right)=f$$

Minkowski problems—recent development

Akman, Andrews, Bianchi, Böröczky, Brendle, Bryan, Chen, Choi, Chow, Cianchi, Cordero-Erausquin, Colesanti, Daskalopoulos, Dou, Feng, Fimiani, Fodor, Fragalà, Gardner, Gluck, Gong, Goodey, Grinberg, Guan, Guang, Haberl, He, Hegedűs, Henk, Hineman, Hong, Hu, Huang, Hug, Ivaki, Jerison, Jian, Jiang, Klain, Klartag, Kolesnikov, Kryvonos, Langharst, Leng, Lewis, Li, Lin, Linke, Liu, Livshyts, Long, Lu, Lutwak, Ma, Marsiglietti, Milman, Miu, Ni, Nyström, Oliker, Pollehn, Rotem, Saari, Salani, Saroglou, Scheuer, Schuster, Sheng, Schneider, Semenov, Stancu, Sun, Trinh, Trudinger, Ulivelli, Umanskiy, Vogel, Wang, Weil, Wu, Xi, Xia, Xie, Xing, Xiong, Xu, Xiao, Yang, Yaskin, Yaskina,Ye, Zhang, Zhou, Zhu, ...

Motivation

Question: Can we do this in Gaussian probability space?

Notation: Gaussian volume $\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K e^{-\frac{|x|^2}{2}} dx$.

What is known in Gaussian space

The Erhard inequality

$$\Phi^{-1}(\gamma_n((1-t)K+tL)) \geq (1-t)\Phi^{-1}(\gamma_n(K))+t\Phi^{-1}(\gamma_n(L)),$$

with "$=$" iff $K = L$. Here,

$$\Phi(x) = \gamma_1((-\infty, x]).$$

Borell, Shenfeld-van Handel...

Dimensional Gaussian Brunn-Minkowski inequality

$$\gamma_n((1-t)K+tL)^{\frac{1}{n}} \geq (1-t)\gamma_n(K)^{\frac{1}{n}}+t\gamma_n(L)^{\frac{1}{n}}$$

for K, L origin-symmetric.

Gardner-Zvavitch 2010
Böröczky, Colesanti, Hosle, Kalantzopoulos, Kolesnikov, Livshyts, Marsiglietti, Nayar, Ritoré, Saroglou, Tkocz, Yepes Nicolás, Zvavitch ...
Eskenazis-Moschidis 2021.

The Gaussian surface area measure

We can define the *Gaussian surface area measure* $S_{\gamma_n,K}$ by

$$\lim_{t\to 0^+} \frac{\gamma_n(K+tL)-\gamma_n(K)}{t} = \int_{S^{n-1}} h_L dS_{\gamma_n,K}.$$

Here, $S_{\gamma_n,K}$ is a finite Borel measure on S^{n-1} given by

$$S_{\gamma_n,K}(\eta) = \frac{1}{(\sqrt{2\pi})^n} \int_{\nu_K^{-1}(\eta)} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x),$$

for each Borel set $\eta \subset S^{n-1}$.

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Problem (The Gaussian Minkowski problem)

Given a finite Borel measure μ on S^{n-1}, when is there a K such that $\mu = S_{\gamma_n,K}$? Is K unique?

The Gaussian Minkowski problem

Let μ be a finite Borel measure on S^{n-1}. Solve

$$\mu = S_{\gamma_n, K}$$

To what extent is the solution unique?

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|\nabla h|^2 + h^2}{2}} \det(h_{ij} + h\delta_{ij}) = f.$$

Gaussian MP v.s. the (classical) MP

- There is an obvious obstruction: Ball and Nazarov showed

$$S_{\gamma_n,K}(S^{n-1}) \lesssim n^{\frac{1}{4}}.$$

  (generalized by Livshyts)
- No translation invariance
- No homogeneity: a variational approach gets you as far as

$$\mu = cS_{\gamma_n,K},$$

  where c comes from the Lagrange multiplier.
- In general, cannot expect uniqueness—$e^{-r^2/2}r^{n-1}$ is not 1-1. Depending on what c is, there are two balls, or 1 ball, or no ball such that

$$S_{\gamma_n,K}(v) = cdv. \qquad (*)$$

  In addition, there *might* be non-ball solutions to (*).

Uniqueness of big solution

Erhard inequality does give partial uniqueness results.

Theorem (Huang-Xi-Z. 2021)

If $S_{\gamma_n,K} = S_{\gamma_n,L}$ and $\gamma_n(K), \gamma_n(L) \geq \frac{1}{2}$, then $K = L$.

Proof

Claim: $\gamma_n(K) = \gamma_n(L)$
Write $\Psi = \Phi^{-1}$. Erhard inequality:

$$\Psi(\gamma_n((1-t)K + tL)) \geq (1-t)\Psi(\gamma_n(K)) + t\Psi(\gamma_n(L)).$$

Differentiating Erhard inequality gives

$$\Psi'(\gamma_n(K)) \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \geq \Psi(\gamma_n(L)) - \Psi(\gamma_n(K)),$$

$$\Psi'(\gamma_n(L)) \int_{S^{n-1}} h_K - h_L dS_{\gamma_n,L} \geq \Psi(\gamma_n(K)) - \Psi(\gamma_n(L)).$$

Using $S_{\gamma_n,K} = S_{\gamma_n,L}$, we have

$$\Psi'(\gamma_n(L)) \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \leq \Psi(\gamma_n(L)) - \Psi(\gamma_n(K)).$$

Hence,

$$\frac{\Psi(\gamma_n(L)) - \Psi(\gamma_n(K))}{\Psi'(\gamma_n(K))} \leq \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \leq \frac{\Psi(\gamma_n(L)) - \Psi(\gamma_n(K))}{\Psi'(\gamma_n(L))}$$

Or

$$\left(\Psi'(\gamma_n(K)) - \Psi'(\gamma_n(L))\right)\left(\Psi(\gamma_n(K)) - \Psi(\gamma_n(L))\right) \leq 0.$$

The function Ψ and Ψ' are strictly increasing on $[1/2, 1]$. Therefore,

$$\gamma_n(K) = \gamma_n(L).$$

□

Erhard inequality:

$$\Psi(\gamma_n((1-t)K+tL)) \geq (1-t)\Psi(\gamma_n(K)) + t\Psi(\gamma_n(L)).$$

Differentiate in t, one gets

$$\Psi'(\gamma_n(K)) \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \geq \Psi(\gamma_n(L)) - \Psi(\gamma_n(K)) = 0,$$

or

$$\int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \geq 0, \text{ with `` = '' iff } K = L.$$

Hence,

$$\int_{S^{n-1}} h_L dS_{\gamma_n,L} = \int_{S^{n-1}} h_L dS_{\gamma_n,K} \geq \int_{S^{n-1}} h_K S_{\gamma_n,K}.$$

Note that the arguments are symmetric in K and L. Therefore, "$=$" holds. □

Existence of large solution

Uniqueness often hints strongly that one can prove existence.

Theorem (Huang-Xi-Z. 2021)

Let μ be an even measure on S^{n-1} that is not concentrated in any subspace and $|\mu| < \frac{1}{\sqrt{2\pi}}$. Then there exists a unique origin-symmetric K with $\gamma_n(K) > 1/2$ such that

$$S_{\gamma_n,K} = \mu.$$

Proof

1. Prove the existence of a smooth solution using degree theory.
2. Approximation to get a weak solution. □

Remark

The symmetry assumption can be removed. $\gamma_n(K) > 1/2$ makes lower C^0 bound trivial.

Uniqueness of solution part 2

The motivation is that when using degree theory to prove the existence of a solution, one only needs to establish the uniqueness of the solution at one point (often when the data is constant).
In dim 2, the Gaussian Minkowski problem for constant data becomes

$$e^{-\frac{h'^2+h^2}{2}}(h''+h)=c. \qquad (**)$$

Theorem (Chen-Hu-Liu-Z. 2023+)

If h is a nonnegative solution to (**), then h has to be a constant solution.

In particular, if $0 < c < e^{-\frac{1}{2}}$, there are precisely two solutions; if $c = e^{-\frac{1}{2}}$, there is exactly one solution; otherwise, there is no solution.
Motivated by (Andrews, 2003)—isotropic curvature flows.

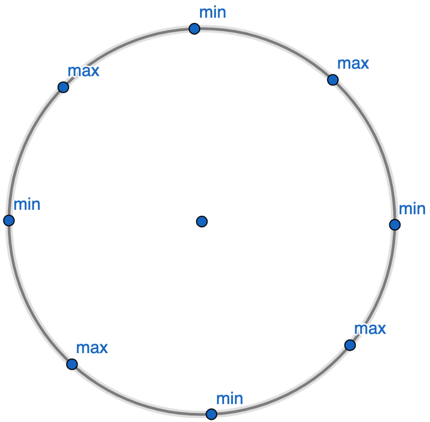
Uniqueness of solution part 2

Assuming h is a non-constant solution.
Idea:

1. Show that critical points of h must be isolated—only finitely many of them.
2. Critical points must be min/max and they alternate.
3. Show that the distance between consecutive critical points only depends on $h_{\min}$ and $h_{\max}$.
4. The distance is represented as an integral Θ and must be π/k for some positive integer k.
5. Show that no such k exists by estimating Θ.

We focus on the case $c \in (0, e^{-\frac{1}{2}})$. The other cases are easier.

Uniqueness of solution part 2



$\Theta = \Delta\theta$ between two consecutive critical pts = const $= \dfrac{2\pi}{2k} = \dfrac{\pi}{k}$

Uniqueness of solution part 2

Lemma

Critical points are isolated.

Proof

By Caffarelli, the solution is smooth.
If θ_i is a sequence of critical points, where $\theta_i \to \theta_0$, then

$$h'(\theta_0) = 0$$
$$h''(\theta_0) = \lim_{i\to\infty} \frac{h'(\theta_i) - h'(\theta_0)}{\theta_i - \theta_0} = 0.$$

Hence, ODE $e^{-\frac{h'^2+h^2}{2}}(h'' + h) = c$ at θ_0 becomes

$$e^{-\frac{h(\theta_0)^2}{2}} h(\theta_0) = c.$$

This has exactly two solutions $h(\theta_0) = m_1$ or $h(\theta_0) = m_2$.

Uniqueness of solution part 2

Hence h solves the following IVP

$$\begin{cases} e^{-\frac{h'^2+h^2}{2}}(h''+h)=c, \\ h(\theta_0)=m, \quad h'(\theta_0)=0, \end{cases}$$

By ODE theory, the solution is unique. But $h \equiv m$ is a solution. This contradicts the fact that h is nonconstant. □

Write $h_0 = \min h$ and $h_1 = \max h$.

Lemma

If θ_* is a critical point, then $h(\theta_*) = h_0$ or h_1.

If h solves

$$e^{-\frac{h'^2+h^2}{2}}(h''+h) = c,$$

then

$$(e^{-\frac{h'^2+h^2}{2}})' = e^{-\frac{h'^2+h^2}{2}}(h''+h)(-h') = -ch',$$

or

$$e^{-\frac{h'^2+h^2}{2}} + ch \equiv E.$$

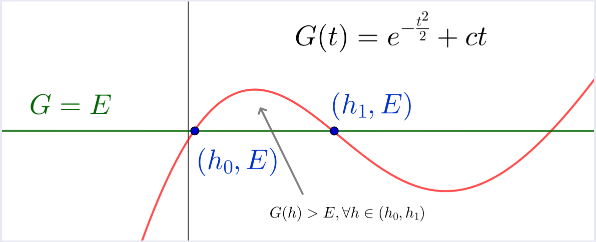
Uniqueness of solution part 2

Proof.

Let $G(t) = e^{-t^2/2} + ct$.
Observation:

$$E = e^{-\frac{h'(\theta)^2 + h(\theta)^2}{2}} + ch(\theta) \leq e^{-\frac{h(\theta)^2}{2}} + ch(\theta) = G(h(\theta)).$$

with equality iff θ is a critical point. In particular, equality holds at θ_*



□

Lemma

If h is a nonconstant solution, then there exists k such that

$$\Theta(h_0, r, c) := \int_0^1 \frac{r}{\sqrt{-(tr + h_0)^2 - 2\log(e^{-\frac{h_0^2}{2}} - ctr)}} dt = \pi/k,$$

where $r = h_1 - h_0$.

Proof

Let $\theta_0 < \theta_1$ be a pair of consecutive critical points. Then

$$\theta_1 - \theta_0 = \int_{\theta_0}^{\theta_1} d\theta = \int_{h_0}^{h_1} \frac{1}{h'(\theta(u))} du = \int_{h_0}^{h_1} \frac{1}{\sqrt{-u^2 - 2\log(E - cu)}} du$$

Here, we used the change of variable $u = h(\theta)$ and the relation

$$e^{-\frac{h'^2 + h^2}{2}} + ch \equiv E.$$

Further making the change of variable $t = (u - h_0)/r$

$$\int_{h_0}^{h_1} \frac{1}{\sqrt{-u^2 - 2\log(E - cu)}} du$$
$$= \int_0^1 \frac{r}{\sqrt{-(h_0 + tr)^2 - 2\log(E - c(h_0 + tr))}} dt.$$

Note that

$$e^{-\frac{h_0^2}{2}} + ch_0 = E.$$

Thus, we get $\Theta(h_0, r, c)$.

Note that for a fixed c, the integral only depends on h_0 and h_1. Since critical points are min/max, one after the other, there exists k such that

$$\Theta(h_0, r, c) = \pi/k.$$

□

Integral estimates

We need to estimate

$$\Theta(h_0, r, c) = \int_0^1 \frac{r}{\sqrt{-(tr + h_0)^2 - 2\log(e^{-\frac{h_0^2}{2}} - ctr)}} dt$$

subject to

1. $G(h_0) = G(h_1)$
2. $h_0 < 1 < h_1$
3. $G'(h_0) > 0$, $G'(h_1) \leq 0$.

The first restriction implies that h_0, r, c have only two degrees of freedom. Fixing two will determine the other one.

Integral estimates

Lemma

- *Fixing c and study* $\Theta = \Theta(r)$, *one has*

$$\liminf_{r\to 0} \Theta(r) \geq \frac{\pi}{\sqrt{1-m_1^2}} > \pi$$

- *Fixing* h_0 *and study* $\Theta = \Theta(r)$, *it is increasing—there is a subtle point about the domain of* r.
- *Fixing* r *and study* $\Theta = \Theta(c)$, *it is increasing—there is a subtle point about the domain of* c.

Uniqueness of solution part 2

- Note that our uniqueness result in dimension 2 does *not* need to assume *a priori* that h is symmetric.

Uniqueness of solution part 2

- Note that our uniqueness result in dimension 2 does *not* need to assume *a priori* that h is symmetric.
- Recently,

Theorem (Ivaki-Milman, 2023+)

If the centroid of K is at the origin and h_K solves

$$e^{-\frac{|\nabla h|^2+h^2}{2}}\det(\nabla^2 h + hI) = c,$$

then h_K has to be a constant solution.

In particular, if K is known to be origin-symmetric, then h has to be a constant solution.

Conjecture

In $n \geq 3$, without any *a priori* assumption on h, is it true that h is a constant solution?

Existence of small solutions in 2d

Theorem (Chen-Hu-Liu-Z. 2023+)

Let $f \in L^1(S^1)$ be an even function such that $\|f\|_{L^1} < \frac{1}{\sqrt{2\pi}}$. If there exists $\tau > 0$ such that $\frac{1}{\tau} < f < \tau$ almost everywhere on S^1, then there exists an origin-symmetric K with $\gamma_2(K) < \frac{1}{2}$ such that

$$dS_{\gamma_2,K}(v) = f(v)dv.$$

Remark

- *Using Ivaki-Milman, this can be done similarly in dimension n.*
- *Here, origin-symmetry is needed for lower bound C^0 estimate. There might be a way to get around this.*