The Minkowski problem in Gaussian probability space

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Motivation

The (classical) Minkowski problem

Let $\mu$ be a finite Borel measure on $S^{n-1}$. Find the necessary and sufficient conditions so that $\mu$ is the surface area measure of a convex body $K$.

To what extent is the solution unique?
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$$S_K(\omega) = \mathcal{H}^{n-1}(\nu_K^{-1}(\omega))$$

Given $v_1, \ldots, v_N \in S^{n-1}$

$$a_1, \ldots, a_N > 0$$
Motivation

Solve

\[ \mu(\cdot) = S_K(\cdot). \]

When \( K \) is \( C^2,^+ \),

\[ S_K(\nu) = \frac{1}{H_{n-1}(K, \nu)} dv. \]

When \( \mu = f dv \), Monge-Ampère equation

\[ \det(h_{ij} + h\delta_{ij}) = f. \]

Minkowski, Aleksandrov, Fenchel, Jessen, Cheng, Yau, Pogorelov, Caffarelli,...
Motivation

Why study *surface area measure*?

Aleksandrov’s variational formula

\[
\left. \frac{d}{dt} \right|_{t=0} V(K + tL) = \int_{S^{n-1}} h_L(v) dS_K(v).
\]

Moral of the story: *surface area measure* is the “derivative” of volume.
Motivation

The Brunn-Minkowski inequality

$$V((1 - t)K + tL)^{\frac{1}{n}} \geq (1 - t)V(K)^{\frac{1}{n}} + tV(L)^{\frac{1}{n}},$$

“=” iff $K$ and $L$ are homothetic.

The Minkowski inequality

$$\frac{1}{n} V_1(K, L) := \frac{1}{n} \int_{S^{n-1}} h_L(v) dS_K(v) \geq V(L)^{\frac{1}{n}} V(K)^{\frac{n-1}{n}},$$

“=” iff $K$ and $L$ are homothetic.

The BM inequality is closely connected to the Minkowski problem.
Motivation

The Minkowski inequality hints
\[ \inf_{h_L} \left\{ \frac{1}{n} \int_{S^{n-1}} h_L(v) d\mu : V(L) = 1 \right\} \]
to solve the Minkowski problem.

- If \( \mu = S_K \) and \( V(L) = 1 \), then the MI states:
  \[ \frac{1}{n} \int_{S^{n-1}} h_L d\mu = \frac{1}{n} \int_{S^{n-1}} h_L dS_K \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}} = V(K)^{\frac{n-1}{n}} \]
  with \( = \) iff \( K \) and \( L \) are homothetic.
Motivation

The Minkowski inequality implies the uniqueness of the solution.

If \( S_K = S_L \), then

\[
V(L) = \frac{1}{n} \int_{S^{n-1}} h_L dS_L = \frac{1}{n} \int_{S^{n-1}} h_L dS_K \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}
\]

Therefore, \( V(L) \geq V(K) \).

Using the symmetry of the above argument, we see equality holds. Hence \( K \) and \( L \) are translations of each other.

Existence and uniqueness of the Minkowski problem also implies the Minkowski inequality with equality condition. (Klain, 2004)
There exists a solution $K$ to the equation

$$\mu = S_K$$

if and only if $\mu$ is not concentrated in any proper subspaces and

$$\int_{S^{n-1}} v d\mu(v) = o.$$ 

The solution is unique up to a translation.
The $L_p$ Minkowski problem (Lutwak, 1993 & 1996)

$$h^{1-p} \det(h_{ij} + h\delta_{ij}) = f$$


The $(L_p)$ dual Minkowski problem (Huang-LYZ 2016):

$$(h^2 + |\nabla h|^2)^{\frac{q-n}{2}} h^{1-p} \det(h_{ij} + h\delta_{ij}) = f$$

The $(L_p)$ chord Minkowski problem (Xi-LYZ 2023):

$$h^{1-p} \tilde{V}_{q-1}(K, Dh) \det(h_{ij} + h\delta_{ij}) = f$$
Minkowski problems—recent development

Question: Can we do this in Gaussian probability space?

Notation: Gaussian volume $\gamma_n(K) = \frac{1}{(\sqrt{2\pi})^n} \int_K e^{-\frac{|x|^2}{2}} dx$. 
What is known in Gaussian space

The Erhard inequality

\[ \Phi^{-1}(\gamma_n((1 - t)K + tL)) \geq (1 - t)\Phi^{-1}(\gamma_n(K)) + t\Phi^{-1}(\gamma_n(L)) \]

with “=” iff \( K = L \). Here,

\[ \Phi(x) = \gamma_1((−\infty, x]) \].

Borell, Shenfeld-van Handel...

Dimensional Gaussian Brunn-Minkowski inequality

\[ \gamma_n((1 - t)K + tL)^{\frac{1}{n}} \geq (1 - t)\gamma_n(K)^{\frac{1}{n}} + t\gamma_n(L)^{\frac{1}{n}} \]

for \( K, L \) origin-symmetric.

Gardner-Zvavitch 2010
Böröczky, Colesanti, Hosle, Kalantzopoulos, Kolesnikov, Livshyts, Marsiglietti, Nayar, Ritoré, Saroglou, Tkocz, Yepes Nicolás, Zvavitch ...
Eskenazis-Moschidis 2021.
The Gaussian surface area measure

We can define the *Gaussian surface area measure* \( S_{\gamma_n,K} \) by

\[
\lim_{t \to 0^+} \frac{\gamma_n(K + tL) - \gamma_n(K)}{t} = \int_{S^{n-1}} h_L dS_{\gamma_n,K}.
\]

Here, \( S_{\gamma_n,K} \) is a finite Borel measure on \( S^{n-1} \) given by

\[
S_{\gamma_n,K}(\eta) = \frac{1}{(\sqrt{2\pi})^n} \int_{\nu_K^{-1}(\eta)} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x),
\]

for each Borel set \( \eta \subset S^{n-1} \).
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for each Borel set $\eta \subset S^{n-1}$.

Problem (The Gaussian Minkowski problem)

*Given a finite Borel measure $\mu$ on $S^{n-1}$, when is there a $K$ such that $\mu = S_{\gamma_n,K}$? Is $K$ unique?*
Let $\mu$ be a finite Borel measure on $S^{n-1}$. Solve

$$\mu = S_{\gamma_n,K}$$

To what extent is the solution unique?

$$\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2} |\nabla h|^2 + h^2} \det(h_{ij} + h\delta_{ij}) = f.$$
There is an obvious obstruction: Ball and Nazarov showed

\[ S_{\gamma n,k}(S^{n-1}) \lesssim n^{\frac{1}{4}}. \]

(generalized by Livshyts)

- No translation invariance
- No homogeneity: a variational approach gets you as far as

\[ \mu = cS_{\gamma n,k}, \]

where \( c \) comes from the Lagrange multiplier.

In general, cannot expect uniqueness—\( e^{-r^2/2}r^{n-1} \) is not 1-1. Depending on what \( c \) is, there are two balls, or 1 ball, or no ball such that

\[ S_{\gamma n,k}(\nu) = cd\nu. \] (*)

In addition, there might be non-ball solutions to (*).
Uniqueness of big solution

Erhard inequality does give partial uniqueness results.

**Theorem (Huang-Xi-Z. 2021)**

If $S_{\gamma n,K} = S_{\gamma n,L}$ and $\gamma_n(K), \gamma_n(L) \geq \frac{1}{2}$, then $K = L$.

**Proof**

Claim: $\gamma_n(K) = \gamma_n(L)$

Write $\Psi = \Phi^{-1}$. Erhard inequality:

$$\Psi(\gamma_n((1 - t)K + tL)) \geq (1 - t)\Psi(\gamma_n(K)) + t\Psi(\gamma_n(L)).$$

Differentiating Erhard inequality gives

$$\Psi'(\gamma_n(K)) \int_{S^{n-1}} h_L - h_K dS_{\gamma n,K} \geq \Psi(\gamma_n(L)) - \Psi(\gamma_n(K)),$$

$$\Psi'(\gamma_n(L)) \int_{S^{n-1}} h_K - h_L dS_{\gamma n,L} \geq \Psi(\gamma_n(K)) - \Psi(\gamma_n(L)).$$
Using $S_{\gamma_n, K} = S_{\gamma_n, L}$, we have

\[
\psi'(\gamma_n(L)) \int_{S^{n-1}} h_L - h_K dS_{\gamma_n, K} \leq \psi(\gamma_n(L)) - \psi(\gamma_n(K)).
\]

Hence,

\[
\frac{\psi(\gamma_n(L)) - \psi(\gamma_n(K))}{\psi'(\gamma_n(K))} \leq \int_{S^{n-1}} h_L - h_K dS_{\gamma_n, K} \leq \frac{\psi(\gamma_n(L)) - \psi(\gamma_n(K))}{\psi'(\gamma_n(L))}
\]

Or

\[
(\psi'(\gamma_n(K)) - \psi'(\gamma_n(L))) (\psi(\gamma_n(K)) - \psi(\gamma_n(L))) \leq 0.
\]

The function $\psi$ and $\psi'$ are strictly increasing on $[1/2, 1]$. Therefore,

\[
\gamma_n(K) = \gamma_n(L).
\]
Erhard inequality:

\[ \psi(\gamma_n((1 - t)K + tL)) \geq (1 - t)\psi(\gamma_n(K)) + t\psi(\gamma_n(L)). \]

Differentiate in \( t \), one gets

\[ \psi'(\gamma_n(K)) \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \geq \psi(\gamma_n(L)) - \psi(\gamma_n(K)) = 0, \]

or

\[ \int_{S^{n-1}} h_L - h_K dS_{\gamma_n,K} \geq 0, \text{ with } "\geq" \text{ iff } K = L. \]

Hence,

\[ \int_{S^{n-1}} h_L dS_{\gamma_n,L} = \int_{S^{n-1}} h_L dS_{\gamma_n,K} \geq \int_{S^{n-1}} h_K S_{\gamma_n,K}. \]

Note that the arguments are symmetric in \( K \) and \( L \). Therefore, "\( = \)" holds.
### Existence of large solution

Uniqueness often hints strongly that one can prove existence.

#### Theorem (Huang-Xi-Z. 2021)

Let $\mu$ be an even measure on $S^{n-1}$ that is not concentrated in any subspace and $|\mu| < \frac{1}{\sqrt{2\pi}}$. Then there exists a unique origin-symmetric $K$ with $\gamma_n(K) > 1/2$ such that

$$S_{\gamma_n, K} = \mu.$$

#### Proof

1. Prove the existence of a smooth solution using degree theory.
2. Approximation to get a weak solution.

#### Remark

The symmetry assumption can be removed. $\gamma_n(K) > 1/2$ makes lower $C^0$ bound trivial.
Uniqueness of solution part 2

The motivation is that when using degree theory to prove the existence of a solution, one only needs to establish the uniqueness of the solution at one point (often when the data is constant).

In dim 2, the Gaussian Minkowski problem for constant data becomes

\[ e^{-\frac{h'^2 + h^2}{2}} (h' + h) = c. \] (**)

**Theorem (Chen-Hu-Liu-Z. 2023+)**

*If h is a nonnegative solution to (**), then h has to be a constant solution.*

In particular, if \( 0 < c < e^{-\frac{1}{2}} \), there are precisely two solutions; if \( c = e^{-\frac{1}{2}} \), there is exactly one solution; otherwise, there is no solution. Motivated by (Andrews, 2003)—isotropic curvature flows.
Uniqueness of solution part 2

Assuming $h$ is a non-constant solution.

Idea:

1. Show that critical points of $h$ must be isolated—only finitely many of them.
2. Critical points must be min/max and they alternate.
3. Show that the distance between consecutive critical points only depends on $h_{\text{min}}$ and $h_{\text{max}}$.
4. The distance is represented as an integral $\Theta$ and must be $\pi/k$ for some positive integer $k$.
5. Show that no such $k$ exists by estimating $\Theta$.

We focus on the case $c \in (0, e^{-\frac{1}{2}})$. The other cases are easier.
Uniqueness of solution part 2

\[ \Theta = \Delta \theta \] between two consecutive critical pts = const = \[ \frac{2\pi}{2k} = \frac{\pi}{k} \].
Uniqueness of solution part 2

Lemma

*Critical points are isolated.*

Proof

By Caffarelli, the solution is smooth. If $\theta_i$ is a sequence of critical points, where $\theta_i \to \theta_0$, then

$$h'(\theta_0) = 0$$

$$h''(\theta_0) = \lim_{i \to \infty} \frac{h'(\theta_i) - h'(\theta_0)}{\theta_i - \theta_0} = 0.$$

Hence, ODE $e^{-\frac{h'^2 + h^2}{2}}(h'' + h) = c$ at $\theta_0$ becomes

$$e^{-\frac{h(\theta_0)^2}{2}} h(\theta_0) = c.$$

This has exactly two solutions $h(\theta_0) = m_1$ or $h(\theta_0) = m_2$. 
Hence $h$ solves the following IVP
\[
\begin{cases}
    e^{-\frac{h'^2 + h^2}{2}} (h'' + h) = c, \\
    h(\theta_0) = m, \quad h'(\theta_0) = 0,
\end{cases}
\]
By ODE theory, the solution is unique. But $h \equiv m$ is a solution. This contradicts the fact that $h$ is nonconstant.
Write $h_0 = \min h$ and $h_1 = \max h$.

Lemma

If $\theta_*$ is a critical point, then $h(\theta_*) = h_0$ or $h_1$.

If $h$ solves

$$e^{-\frac{h'^2 + h^2}{2}} (h'' + h) = c,$$

then

$$(e^{-\frac{h'^2 + h^2}{2}})' = e^{-\frac{h'^2 + h^2}{2}} (h'' + h)(-h') = -ch',$$

or

$$e^{-\frac{h'^2 + h^2}{2}} + ch \equiv E.$$
Uniqueness of solution part 2

Proof.

Let $G(t) = e^{-t^2/2} + ct$.

Observation:

$$E = e^{-\frac{h'(\theta)^2 + h(\theta)^2}{2}} + ch(\theta) \leq e^{-\frac{h(\theta)^2}{2}} + ch(\theta) = G(h(\theta)).$$

with equality iff $\theta$ is a critical point. In particular, equality holds at $\theta^*$. 

$$G(t) = e^{-\frac{t^2}{2}} + ct$$
Lemma

If $h$ is a nonconstant solution, then there exists $k$ such that

$$
\Theta(h_0, r, c) := \int_0^1 \frac{r}{\sqrt{-(tr + h_0)^2 - 2 \log(e^{-\frac{h_0^2}{2}} - c tr)}} dt = \pi/k,
$$

where $r = h_1 - h_0$.

Proof

Let $\theta_0 < \theta_1$ be a pair of consecutive critical points. Then

$$
\theta_1 - \theta_0 = \int_{\theta_0}^{\theta_1} d\theta = \int_{h_0}^{h_1} \frac{1}{h'(\theta(u))} du = \int_{h_0}^{h_1} \frac{1}{\sqrt{-u^2 - 2 \log(E - cu)}} du
$$

Here, we used the change of variable $u = h(\theta)$ and the relation

$$
e^{-\frac{h'^2 + h^2}{2}} + ch \equiv E.$$

Further making the change of variable \( t = (u - h_0)/r \)

\[
\int_{h_0}^{h_1} \frac{1}{\sqrt{-u^2 - 2 \log(E - cu)}} du
= \int_0^1 \frac{r}{\sqrt{-(h_0 + tr)^2 - 2 \log(E - c(h_0 + tr))}} dt.
\]

Note that

\[ e^{-\frac{h_0^2}{2}} + ch_0 = E. \]

Thus, we get \( \Theta(h_0, r, c) \).

Note that for a fixed \( c \), the integral only depends on \( h_0 \) and \( h_1 \). Since critical points are min/max, one after the other, there exists \( k \) such that

\[ \Theta(h_0, r, c) = \pi/k. \]
Integral estimates

We need to estimate

$$\Theta(h_0, r, c) = \int_0^1 \frac{r}{\sqrt{-(tr + h_0)^2 - 2 \log(e^{-\frac{h_0^2}{2}} - ctr)}} dt$$

subject to

1. $G(h_0) = G(h_1)$
2. $h_0 < 1 < h_1$
3. $G'(h_0) > 0, G'(h_1) \leq 0$.

The first restriction implies that $h_0, r, c$ have only two degrees of freedom. Fixing two will determine the other one.
Integral estimates

Lemma

Fixing $c$ and study $\Theta = \Theta(r)$, one has

$$\liminf_{r \to 0} \Theta(r) \geq \frac{\pi}{\sqrt{1 - m^2_1}} > \pi$$

Fixing $h_0$ and study $\Theta = \Theta(r)$, it is increasing—there is a subtle point about the domain of $r$.

Fixing $r$ and study $\Theta = \Theta(c)$, it is increasing—there is a subtle point about the domain of $c$. 
Note that our uniqueness result in dimension 2 does \textit{not} need to assume \textit{a priori} that $h$ is symmetric.
Uniqueness of solution part 2

- Note that our uniqueness result in dimension 2 does \textit{not} need to assume \textit{a priori} that $h$ is symmetric.
- Recently,

\begin{theorem}[Ivaki-Milman, 2023+]
If the centroid of $K$ is at the origin and $h_K$ solves
\[ e^{-\frac{|\nabla h|^2 + h^2}{2}} \det(\nabla^2 h + hl) = c, \]
then $h_K$ has to be a constant solution.
\end{theorem}

In particular, if $K$ is known to be origin-symmetric, then $h$ has to be a constant solution.

\begin{conjecture}
In $n \geq 3$, without any \textit{a priori} assumption on $h$, is it true that $h$ is a constant solution?
\end{conjecture}
Existence of small solutions in 2d

Theorem (Chen-Hu-Liu-Z. 2023+)

Let \( f \in L^1(S^1) \) be an even function such that \( \|f\|_{L^1} < \frac{1}{\sqrt{2\pi}} \). If there exists \( \tau > 0 \) such that \( \frac{1}{\tau} < f < \tau \) almost everywhere on \( S^1 \), then there exists an origin-symmetric \( K \) with \( \gamma_2(K) < \frac{1}{2} \) such that

\[
dS_{\gamma_2,K}(v) = f(v)dv.
\]

Remark

- Using Ivaki-Milman, this can be done similarly in dimension \( n \).
- Here, origin-symmetry is needed for lower bound \( C^0 \) estimate. There might be a way to get around this.