On complemented Brunn-Minkowski type inequalities

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(joint work with A. Zvavitch)

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The Brunn-Minkowski inequality for measures

Theorem (Borell-Brascamp&Lieb)

Let μ be an absolutely continuous measure on $\mathbb{R}^n$ associated to a p-concave density, with $p \in [-1/n, \infty]$.
Let $K, L \subset \mathbb{R}^n$ be (non-empty) compact sets with $\mu(K)\mu(L) > 0$ and let $\lambda \in (0, 1)$. Then

$$\mu\big((1-\lambda)K + \lambda L\big) \geq \big((1-\lambda)\mu(K)^q + \lambda\mu(L)^q\big)^{1/q},$$

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The Brunn-Minkowski inequality for $\mathrm{vol}(\cdot)$

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$\left(\mathrm{d}\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} e^{\frac{-|x|^2}{2}}\mathrm{d}x\right)$

Brunn-Minkowski for the Gaussian measure $\gamma_n(\cdot)$

$$\gamma_n((1-\lambda)K + \lambda L) \geq \gamma_n(K)^{1-\lambda}\gamma_n(L)^{\lambda}.$$

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$$\mathrm{vol}\big((1-\lambda)K \widetilde{+} \lambda L\big)^{1/n} \leq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}.$$

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We will say that K is a *generalized star body* if K is a starshaped set with continuous radial function ρ_K (on its support), but we do not impose it to be positive and finite, i.e., we allow that $0 \leq \rho_K(u) \leq \infty$ for any $u \in \mathbb{S}^{n-1}$.

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The above result holds true for generalized star bodies K and L.

Complemented Brunn-Minkowski inequalities

Given a q-concave measure on $\mathbb{R}^n$, namely, a measure satisfying

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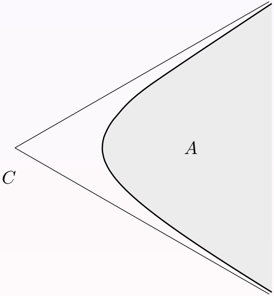
This problem was initiated and studied by E. Milman and L. Rotem in 2014, and they obtained various interesting properties on the class of q-complemented measures and that these inequalities hold for measures associated to p-homogeneous densities.

Theorem (Schneider (2018))

Let $C \subset \mathbb{R}^n$ be a closed convex cone with interior points, let $A, B \subset C$ be closed convex sets s.t. $0 < \mathrm{vol}(C \setminus A), \mathrm{vol}(C \setminus B) < \infty$, and let $\lambda \in (0, 1)$.



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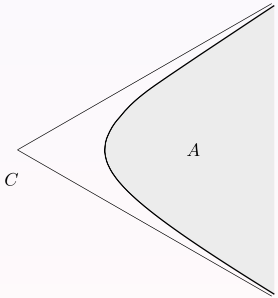

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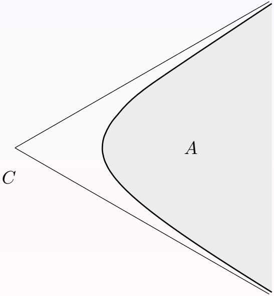

Schneider's proof adapts the classical Kneser-Süss approach to the Brunn-Minkowski inequality for convex bodies, but needs extra steps.

Dual vs Complemented Brunn-Minkowski inequalities

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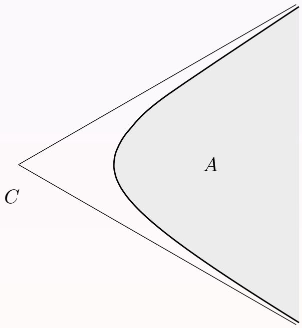

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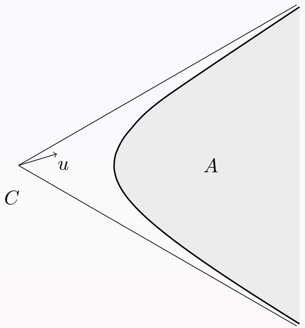
Given a set $A \subset \mathbb{R}^n$,



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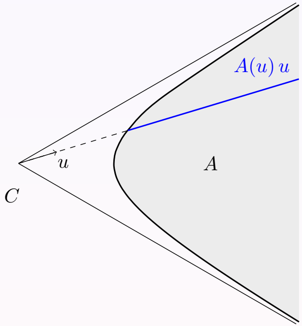
Given a set $A \subset \mathbb{R}^n$, we write

$$A(u) := \{t \in \mathbb{R}_{\geq 0} : tu \in A\}$$

for any $u \in \mathbb{S}^{n-1}$.



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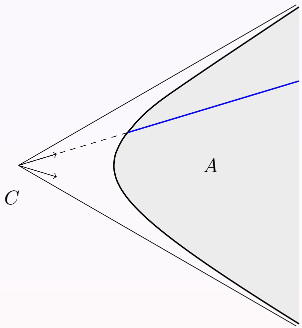

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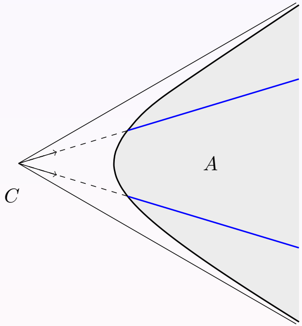

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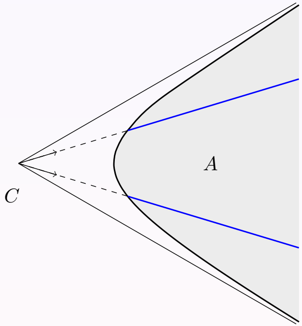

Moreover, for any $A, B \subset \mathbb{R}^n$, we define

$$(A \widetilde{+} B)(u) := \begin{cases} A(u) + B(u) & \text{if both } A(u), B(u) \text{ are non-empty,} \\ \emptyset & \text{otherwise.} \end{cases}$$

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Now, we then extend the radial sum $\widetilde{+}$ to arbitrary sets as follows:

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Note also that, given a closed convex cone C with interior points, if $A, B \subset C$ are convex then

$$C \setminus \big(A\widetilde{+}B\big) = \big(C\setminus A\big)\widetilde{+}\big(C\setminus B\big).$$

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Theorem (Y.N., Zvavitch (2023+))

Let $C \subset \mathbb{R}^n$ be a closed convex cone with interior points, let $A, B \subset C$ be Borel sets s.t. $0 < \mathrm{vol}(C \setminus A), \mathrm{vol}(C \setminus B) < \infty$, and let $\lambda \in (0, 1)$.

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Proof (complemented dual B-M for coconvex sets)

From

$$C \setminus \left(A \widetilde{+} B\right) = \left(C \setminus A\right) \widetilde{+} \left(C \setminus B\right)$$

we clearly have that

$$C \setminus \left((1-\lambda)A \widetilde{+} \lambda B\right) = \Big(C \setminus \left((1-\lambda)A\right)\Big) \widetilde{+} \Big(C \setminus (\lambda B)\Big)$$

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we clearly have that

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If equality holds then we have equality in the dual Brunn-Minkowski inequality for the sets $K = C \setminus A$ and $L = C \setminus B$, and thus $C \setminus A = \alpha(C \setminus B)$ for some $\alpha > 0$, which is equivalent to the identity $A = \alpha B$.

Complemented Gaussian Brunn-Minkowski inequalities

When considering the Gaussian measure $\gamma_n(\cdot)$, the (in some sense) 'real concavity' that one naturally has is provided by the well-known Ehrhard inequality:

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Let $A, B \subset \mathbb{R}^n$ be Borel sets. Then

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By using the above extension of the radial sum, we showed the following:

Theorem (Y.N., Zvavitch (2023+))

Let $C \subset \mathbb{R}^n$ be a closed convex cone with interior points and let $A, B \subset C$ be Borel sets. Then

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This result recovers (as before in the case of the volume) the previously known Gaussian dual Brunn-Minkowski inequality.

On complemented Brunn-Minkowski type inequalities

J. Yepes Nicolás

Universidad de Murcia

(joint work with A. Zvavitch)

INdAM Meeting

CONVEX GEOMETRY - ANALYTIC ASPECTS

Cortona

June 27th, 2023