Stability of polydisc slicing

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joint work with N. Glover and T. Tkocz

Notation

- Consider $\mathbb{R}^n$ with the standard inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and let $|x| = \sqrt{\langle x, x\rangle}$.
- Let $S^{n-1} = \{x \in \mathbb{R}^n : \ |x| = 1\}$
- We write $e_1, e_2, \dots, e_n$ for the standard basis vectors.
- For $a \in S^{n-1}$ we define the hyperplane perpendicular to a

$$a^{\perp} = \{x \in \mathbb{R}^n : \ \langle x, a\rangle = 0\}.$$

- vol_i stands for a Lebesgue measure on an appropriate subspace of dimension i.

Extremal sections of the cube

Let $Q_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ be the cube in $\mathbb{R}^n$.

Theorem (Ball)

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$$1 \leq \mathrm{vol}_{n-1}(Q_n \cap a^{\perp}) \leq \sqrt{2}$$

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  for any $i = 1, \ldots, n$.
- The upper bound due to Ball ('86), uniquely attained at

$$a = e_i \pm e_j$$

  for any $i \neq j$.

Stability of cube slicing

Theorem (Chasapis, Nayar, Tkocz)

For every unit vector a in $\mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have

$$1 + \frac{1}{54}|a - e_1|^2 \leq \mathrm{vol}_{n-1}(Q_n \cap a^{\perp}) \leq \sqrt{2} - 6 \cdot 10^{-5} \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|$$

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- The above formulation is due to Chasapis, Nayar and Tkocz ('22) (constant in the upper bound is due to Eskenazis, Nayar and Tkocz)
- A local version was established by Melbourne and Roberto ('22) (with applications to information theory)

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- For $z, w \in \mathbb{C}^n$, we let as usual $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$ be their standard inner product, and let $|z| = \sqrt{\langle z, \bar{z}\rangle}$.
- Let $\mathbb{D}$ be the unit disc in the complex plane and let

$$\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D} = \{z \in \mathbb{C}^n : \max_{1 \le j \le n} |z_j| \le 1\}$$

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  be the polydisc in $\mathbb{C}^n$.
- In this setting, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the standard embedding.
- Note that here for $a \in \mathbb{C}$, the subspace $a^\perp$, as a subspace of $\mathbb{R}^{2n}$, is of real dimension $2n - 2$.
- Note that $\mathrm{vol}_{2n-2}(\mathbb{D}^{n-1}) = \pi^{n-1}$
  (obtained as the canonical section $\mathbb{D}^n \cap (1, 0, \ldots, 0)^\perp$).

Extremal section of the polydisc

A counterpart of Ball's cube slicing in $\mathbb{C}^n$ was developed by Oleszkiewicz and Pełczyński ('00):

Theorem (Oleszkiewicz, Pełczyński)

For every (complex) hyperplane $a^\perp = \{z \in \mathbb{C}^n : \ \langle z, a \rangle = 0\}$ *orthogonal to the vector* a *in* $\mathbb{C}^n$*, we have*

$$1 \leq \frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) \leq 2.$$

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- The lower bound is attained uniquely at hyperplanes orthogonal to the "standard basis vectors"

$$\xi e_j, \quad 1 \leq j \leq n, \text{ and } \xi \in \mathbb{C} \text{ s.th } |\xi| = 1.$$

- the upper one is attained uniquely at hyperplanes orthogonal to the vectors

$$e_j + \xi e_k, \quad 1 \leq j < k \leq n, \text{ and } \xi \in \mathbb{C} \text{ s.th } |\xi| = 1.$$

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$$1 \leq \frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) \leq 2.$$

Thanks to the symmetries of $\mathbb{D}^n$ under the permutations of the coordinates as well as complex rotations along axes

$$z \mapsto (e^{it_1} z_1, \dots, e^{it_n} z_n),$$

it suffices to consider real nonnegative vectors with nonincreasing components.

Stability of polydisc slicing

Theorem (Glover, Tkocz, W.)

For $n \geq 2$ and every unit vector a in $\mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have

$$\frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^\perp) \leq 2 - \min\left\{ 10^{-40} \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|, \ \frac{1}{76} \sum_{j=1}^{n} a_j^4 \right\}.$$

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The upper bound is the minimum over two quantities: the distance to the *unique* extremiser and the ℓ_4 norm of a. The latter appears to account for the fact that

$$\lim_{n \to \infty} \frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}\left(\mathbb{D}^n \cap \left(\tfrac{1}{\sqrt{n}}, \ldots, \tfrac{1}{\sqrt{n}} \right)^{\perp} \right) = 2.$$

In other words, polidysc slicing admits an additional *asymptotic* (Gaussian) extremiser.

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In the real case,

$$\lim_{n\to\infty} \mathrm{vol}_{n-1}\left(\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]^n \cap \left(\tfrac{1}{\sqrt{n}}, \ldots, \tfrac{1}{\sqrt{n}} \right)^\perp \right) = \sqrt{\frac{6}{\pi}} < \sqrt{2}.$$

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A stability for the lower-bound can be easily extracted from previous works of Chasapis, Nayar, Tkocz:

$$\frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^{\perp}) \geq 1 + \frac{1}{4} |a - e_1|^2.$$

Sketch of the proof

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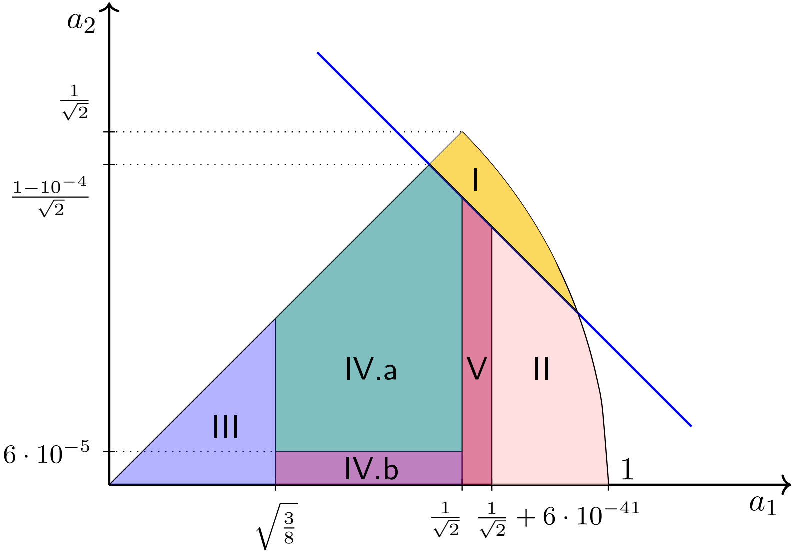

For convenience, we consider the normalised section function

$$A_n(a) = \frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^{\perp}), \qquad a \in \mathbb{R}^n$$

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Our approach, to a large extent, relies on the following probabilistic formula for the volume of sections of the polydisc:

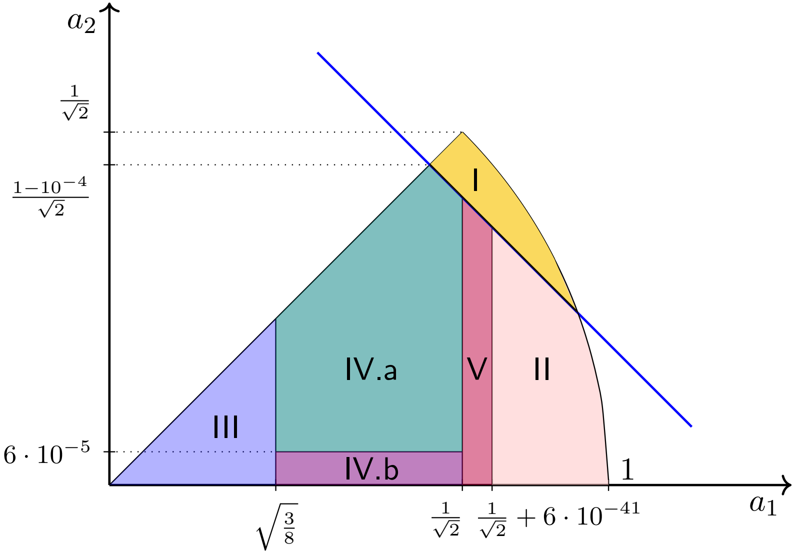
Lemma (Brzezinski ('13))

For every $n \geq 1$ and every **unit** *vector a in $\mathbb{R}^n$, we have*

$$A_n(a) = \mathbb{E} \left| \sum_{k=1}^{n} a_k \xi_k \right|^{-2},$$

where $\xi_1, \xi_2, \ldots$ are independent random vectors uniform on the unit sphere S^3 in $\mathbb{R}^4$.

Sketch of the proof: I



Sketch of the proof: I

We set

$$\delta(a) = \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|^2 = 2 - \sqrt{2}(a_1 + a_2)$$

We reapply polydisc slicing in a lower dimension to a portion of a, which yields its self-improvement and gives a quantitative deficit:

Lemma

We have, $A_n(a) \leq 2 - \frac{1}{25}\sqrt{\delta(a)}$, provided that $\delta(a) \leq 2 \cdot 10^{-4}$.

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Let X and Y be independent rotationally invariant random vectors in $\mathbb{R}^4$,

$$\mathbb{E}|X+Y|^{-2} = \mathbb{E}\min\{|X|^{-2}, |Y|^{-2}\}$$

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Let $X = a_1\xi_1 + a_2\xi_2$ and $Y = \sum_{j=3}^{n} a_j\xi_j$. Then,

$$A_n(a) = \mathbb{E}\min\{|X|^{-2}, |Y|^{-2}\} \leq \mathbb{E}_X \min\{|X|^{-2}, \mathbb{E}_Y|Y|^{-2}\},$$

where we use the concavity of $t \mapsto \min\{\alpha, t\}$.

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By polydisc slicing, $\mathbb{E}_Y|Y|^{-2} \leq \frac{2}{1-a_1^2-a_2^2}$. Hence

$$\begin{aligned} A_n(a) &\leq \mathbb{E}\min\left\{|X|^{-2}, \frac{2}{1-a_1^2-a_2^2}\right\} \\ &= \mathbb{E}|X|^{-2} - \mathbb{E}\left(|X|^{-2} - \frac{2}{1-a_1^2-a_2^2}\right)_+ \end{aligned}$$

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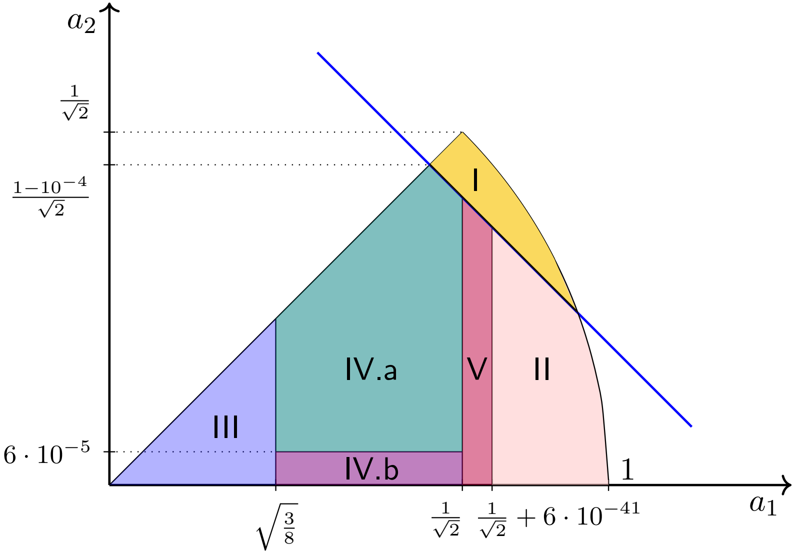
Use again that $\mathbb{E}|X|^{-2} = \min\{a_1^{-2}, a_2^{-2}\} = a_1^{-2}$.

Sketch of the proof: II



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Recall that

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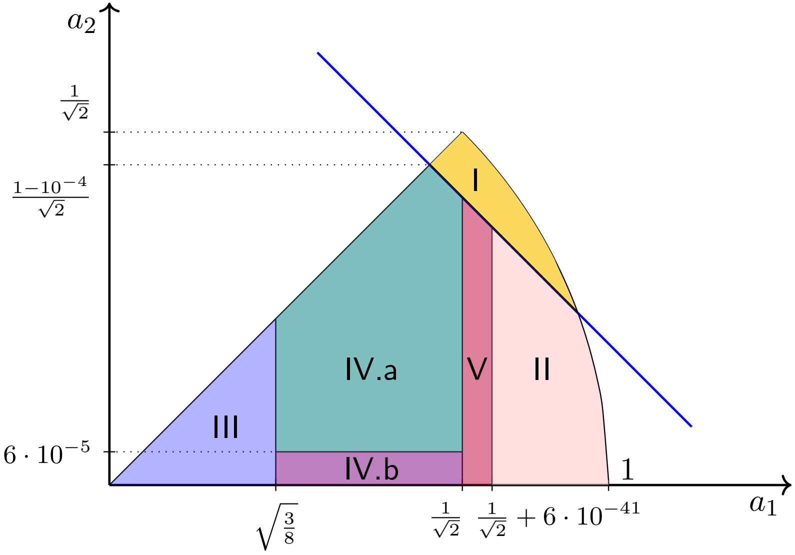
Let $X = a_1 \xi_1$ and $Y = \sum_{i=2}^{n} a_i \xi_i$.

$$A_n(a) = \mathbb{E} \min\{|X|^{-2}, |Y|^{-2}\} \leq a_1^{-2} \leq 2(1 - 6\sqrt{2} \cdot 10^{-41}).$$

Sketch of the proof: III



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For every $n \geq 1$ and every *unit* vector a in $\mathbb{R}^n$, we have

$$A_n(a) \leq 2 \prod_{k=1}^{n} \Psi(a_k^{-2})^{a_k^2},$$

where for $s > 0$, we define $\Psi(s) = \frac{s}{4} \int_0^\infty \left| \frac{2J_1(t)}{t} \right|^s t dt$

where $J_1(t) = \frac{t}{2} \sum_{k=0}^\infty \frac{(-1)^k}{2^{2k} k!(k+1)!} t^{2k}$ is the Bessel function of order 1.

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Lemma. For the function Ψ defined above, we have

$$\Psi(s) \leq \begin{cases} 1 - \frac{1}{12}(s-2)^2, & 2 \leq s \leq \frac{8}{3}, \\ 1 - \frac{1}{151s}, & s > \frac{8}{3}. \end{cases}$$

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Lemma

We have that $A_n(a) \leq 2\exp\left\{-\frac{1}{151}\|a\|_4^4\right\}$*, provided that* $a_1 \leq \sqrt{\frac{3}{8}}$*.*

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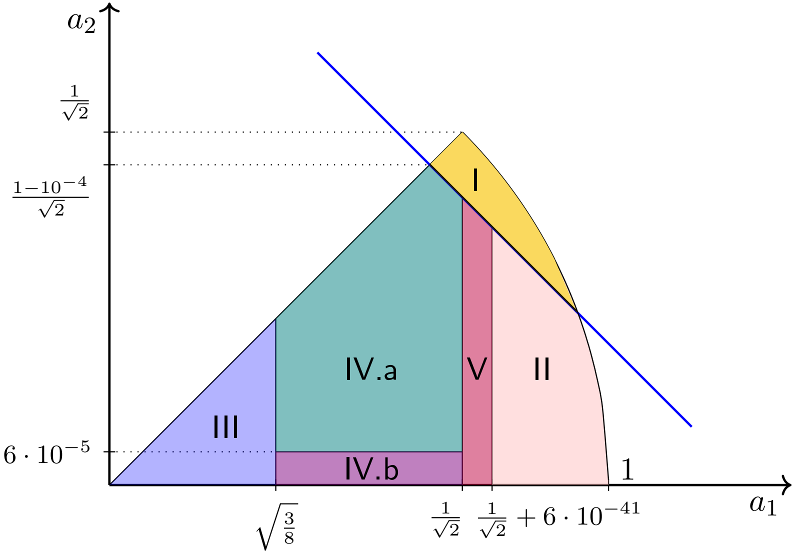
By the assumption, $a_k^{-2} \geq \frac{8}{3}$ for all k, thus

$$\begin{aligned} A_n(a) &\leq 2\prod_{k=1}^{n} \Psi(a_k^{-2})^{a_k^2} \\ &\leq 2\prod_{k=1}^{n} \big(1 - \tfrac{1}{151} a_k^2\big)^{a_k^2} \\ &\leq 2\exp\Big\{-\tfrac{1}{151}\sum_{k=1}^{n} a_k^4\Big\}. \end{aligned}$$

Sketch of the proof: IV



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$A_n(a) \leq 2 - 10^{-19}$, if $\sqrt{\frac{3}{8}} \leq a_1 \leq \frac{1}{\sqrt{2}}$ and $6 \cdot 10^{-5} \leq a_2 \leq \frac{1-10^{-5}}{\sqrt{2}}$.

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Recall that:

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Oleszkiewicz and Pełczyński's approach crucially relies on the fact that

$$\sup_{s \geq 2} \Psi(s) = 1.$$

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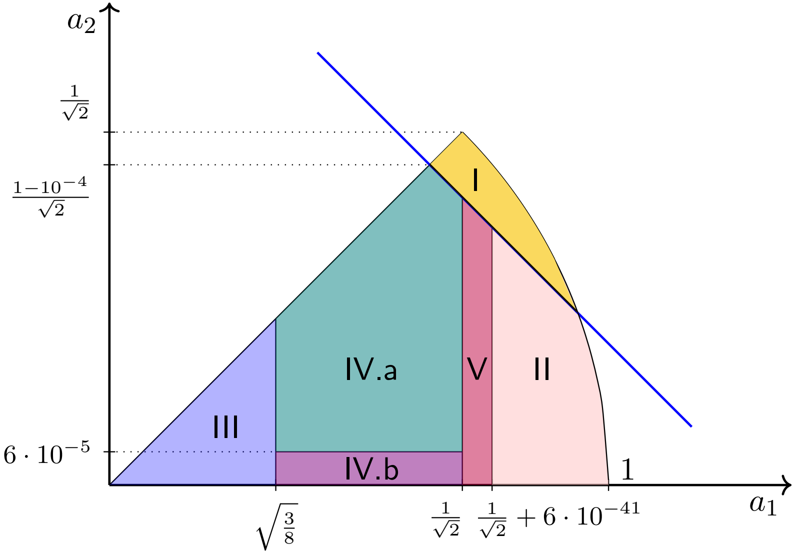
Hence, $\Psi(a_k^{-2}) \le 1$ for each k, since $a_k^{-2} \ge 2$ for each k. Using this (for all k except $k = 2$)

$$A_n(a) \le 2 \prod_{k=1}^{n} \Psi(a_k^{-2})^{a_k^2} \le 2\Psi(a_2^{-2})^{a_2^2}$$

Sketch of the proof: IV.b



Sketch of the proof: IV.b

When $a_1 \approx \frac{1}{\sqrt{2}}$ but $a_2, \ldots, a_n$ are small, we will employ a Berry-Esseen type bound with an explicit constant for random vectors in $\mathbb{R}^4$.

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Raič ('19) has obtained such a result for an arbitrary dimension.

Theorem (Raič)

Let $X_1, \ldots, X_n$ be independent mean 0 random vectors in $\mathbb{R}^d$ such that $\sum_{j=1}^n X_j$ has the identity covariance matrix. Let G be a standard Gaussian random vector in $\mathbb{R}^d$. Then

$$\sup_A \left| \mathbb{P}\Big(\sum_{j=1}^n X_j \in A \Big) - \mathbb{P}\big(G \in A \big) \right| \leq (42 d^{1/4} + 16) \sum_{j=1}^n \mathbb{E}|X_j|^3,$$

where the supremum is over all Borel convex sets in $\mathbb{R}^d$.

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Lemma

$A_n(a) \leq 2 - 10^{-5}$, *provided that* $\sqrt{\frac{3}{8}} \leq a_1 \leq \frac{1}{\sqrt{2}}$ *and* $a_2 \leq 6 \cdot 10^{-5}$.

Let $Y = \sum_{j=2}^{n} a_j \xi_j$ and observe that

$$A_n(a) = \mathbb{E}\left|a_1\xi_1 + Y\right|^{-2} = \mathbb{E}\min\left\{a_1^{-2}, |Y|^{-2}\right\}$$

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Since Y has covariance matrix $\frac{1-a_1^2}{4}\mathrm{Id}$, using the Berry-Esseen bound (applied to $d = 4$ and $X_j = \frac{2}{\sqrt{1-a_1^2}} a_j \xi_j,\ \ j = 2, \ldots, n$).

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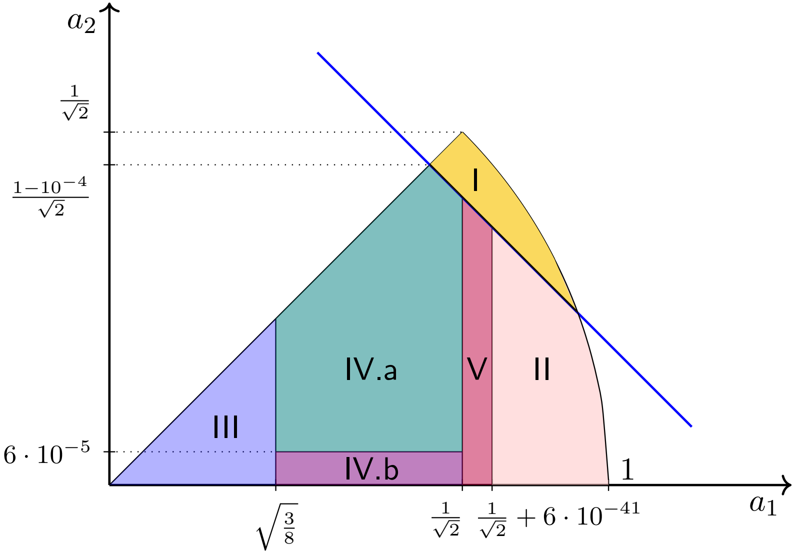
$$\mathbb{P}\big(|Y|^{-2} > t\big) \le \mathbb{P}\Big(\Big(\sqrt{\tfrac{1-a_1^2}{4}}|G|\Big)^{-2} > t\Big) + (42\sqrt{2}+16)\sum_{j=2}^{n}\mathbb{E}\left|\frac{2}{\sqrt{1-a_1^2}}a_j\xi_j\right|^3$$

Sketch of the proof: V



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Theorem (Koldobsky-Paouris-Zymonopoulou ('13))

Let K be a complex symmetric convex body K in $\mathbb{C}^n$, that is K is a convex body in $\mathbb{R}^{2n}$ with $e^{it}z \in K$, whenever $z \in K$, $t \in \mathbb{R}$. Then the function

$$z \mapsto \frac{|z|}{(\mathrm{vol}_{2n-2}(K \cap z^{\perp}))^{1/2}}$$

defines a norm on $\mathbb{C}^n$.

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Lemma

For unit vectors a, b in $\mathbb{R}^n$, we have

$$|A_n(a) - A_n(b)| \leq 4\sqrt{2}|a - b|.$$

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Lemma

We have, $A_n(a) \leq 2 - 10^{-20}$, *provided that* $\frac{1}{\sqrt{2}} < a_1 \leq \frac{1}{\sqrt{2}} + 6 \cdot 10^{-41}$ *and* $a_2 \leq \frac{1-10^{-4}}{\sqrt{2}}$.

We consider the following modification of a, the vector

$$b = \left(\frac{1}{\sqrt{2}}, \sqrt{a_1^2 + a_2^2 - \frac{1}{2}}, a_3, \ldots, a_n \right).$$

Then

$$A_n(a) \leq A_n(b) + 4\sqrt{2}|a - b| \leq 2 - 10^{-19} + 8|a - b|.$$

Stability of polydisc slicing

Theorem (Glover, Tkocz, W.)

For $n \geq 2$ and every unit vector a in $\mathbb{R}^n$ with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, we have

$$\frac{1}{\pi^{n-1}} \mathrm{vol}_{2n-2}(\mathbb{D}^n \cap a^{\perp}) \leq 2 - \min\left\{ 10^{-40} \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|, \ \frac{1}{76} \sum_{j=1}^{n} a_j^4 \right\}.$$

Thank you for your attention!