On a generalization of the Alexandrov–Fenchel inequality

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Convex Geometry - Analytic aspects
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Alexandrov–Fenchel inequality

$$V(K,L,C_1,\ldots,C_{d-2})^2 \geq V(K,K,C_1,\ldots,C_{d-2})V(L,L,C_1,\ldots,C_{d-2})$$

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- generalization of the isoperimetric inequality
- implies many inequalities
- connection with complex algebraic geometry/Kähler geometry

Notation

$$f(x) = \sum_{|\alpha|=d} c_\alpha x^\alpha, \quad x \in \mathbb{R}^n$$

$\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n = d$

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For $K_1, \ldots, K_n$ in $\mathbb{R}^d$ define

$$V(K^\alpha) = V(K_1[\alpha_1], \ldots, K_n[\alpha_n]), \quad |\alpha| = d$$

and

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If $f = gh$ write

$$V(g(K), h(K)) = V(f(K)).$$

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$$\begin{aligned} V(f(K)) &= V(K_1, K_1) + 2V(K_1, K_2) + V(K_2, K_2) \\ &= V(K_1 + K_2, K_1 + K_2) \end{aligned}$$

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A generalization of the Alexandrov–Fenchel inequality

Corollary (Ross–Süss–W. ’23)
Let $K, L, C_1, \dots, C_n$ be convex bodies in $\mathbb{R}^d$.

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Examples.

1. $\lambda = (1, \ldots, 1) \in \mathbb{N}^{d-2}$, $s_\lambda(x_1, \ldots, x_{d-2}) = x_1 \cdots x_{d-2}$.
2. $\lambda = (2)$, $s_\lambda(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$.

Hodge–Riemann bilinear relations for Schur classes

Theorem (Ross–Toma 2022)
Let M be an d-dimensional Kähler manifold. Let $s_\lambda(x_1,\ldots,x_n)$ be a non-zero Schur polynomial of degree $d-2$. Let $\omega_1,\ldots,\omega_n$ be Kähler forms on M. Then the quadratic form

$$Q\colon H^{1,1}_{\mathbb{R}}(M) \to \mathbb{R}, \quad Q([\alpha]) = \int_M \alpha^2 \wedge s_\lambda(\omega_1,\ldots,\omega_n)$$

has signature $(+,-,\ldots,-)$.

Lorentzian polynomials

Log-concave sequences, Kähler package, and Lorentzian polynomials

A sequence of non-negative numbers $a_0, \ldots, a_d$ is *log-concave* if

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”A log-concave sequence points to the existence of a Kähler package in the background” (June Huh)

1. Poincaré duality
2. hard Lefschetz theorem
3. Hodge–Riemann bilinear relations

Lorentzian polynomials

$\mathrm{H}_n^d \subset \mathbb{R}[x_1, \ldots, x_n]$ homogeneous polynomials

Definition (Brändén–Huh ’20)
$f \in \mathrm{H}_n^d$ is *strictly Lorentzian* if for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = d - 2$ the quadratic form $\partial^\alpha f$ has positive coefficients and signature $(+, -, \ldots, -)$.

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Limits of strictly Lorenztian polynomials are called Lorentzian.

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3. Define
$$\mathrm{N}(x^\alpha) = \frac{1}{\alpha!} x^\alpha = \frac{1}{\alpha_1! \cdots \alpha_n!} x^\alpha$$
   Then $N(s_\lambda)$ is Lorentzian (Huh–Matherne–Mészáros–Dizier ’22)

Dually Lorentzian polynomials

$\mathbb{R}_\kappa[x_1, \ldots, x_n]$ polynomials of degree at most κ_i in x_i.

Definition (Ross–Süss–W. '23)
$s \in \mathbb{R}_\kappa[x_1, \ldots, x_n]$ is *dually Lorentzian* if

$$s^\vee(x_1, \ldots, x_n) = \mathrm{N}(x^\kappa s(x_1^{-1}, \cdots, x_n^{-1}))$$

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2. Covolume polynomials of irred varieties in $\mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_k}$ (Aluffi ’23)

Theorem (Ross–Süss–W. ’23)
$s \neq 0$ is dually Lorentzian if and only if the operator

$$\partial_s = s(\partial_1, \ldots, \partial_n) \colon \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$$

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Theorem (Ross–Süss–W. ’23)
Let A be an $n \times m$ matrix with non-negative entries. If $s \in \mathbb{R}[x_1, \ldots, x_n]$ is dually Lorentzian, then so is $s(Ay)$, $y \in \mathbb{R}^m$.

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for every non-zero dually Lorentzian polynomial s of degree $d - 2$.

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Equality holds if and only if p is proportional to q.

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$$\left(\int_Y \alpha\beta s(\omega_1,\ldots,\omega_n)\right)^2 \geq \int_Y \alpha^2 s(\omega_1,\ldots,\omega_n) \int_Y \beta^2 s(\omega_1,\ldots,\omega_n)$$

and equality holds if and only if α is proportional to β.

Proofs

Operations preserving Lorentzian polynomials

Theorem (Brändén–Huh ’20)
Let A be an $n \times m$ matrix with non-negative entries. If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Lorentzian, then so is $f(Ay)$, $y \in \mathbb{R}^m$.

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Theorem (Brändén–Huh ’20)
Let $T\colon \mathbb{R}_\kappa[x_i] \to \mathbb{R}_\gamma[x_i]$ be linear and homogeneous. If the *symbol*

$$\mathrm{sym}_T(x, y) = \sum_\beta \binom{\kappa}{\beta} T(x^\beta) y^{\kappa-\beta} \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

is Lorentzian, then T preserves Lorentzian polynomials.

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Hence is $\operatorname{sym}_{\partial_s}$ Lorentzian if $s^\vee$ is Lorentzian.

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is Lorentzian. Hence

$$\partial_s f(0,y) = \frac{d!}{2} V((y_1K + y_2L)^2, s(C_1, \ldots, C_n))$$

is Lorentzian.

Proof of equality cases

$$\left(\int_Y \alpha\beta s(\omega_1,\ldots,\omega_n)\right)^2 \geq \int_Y \alpha^2 s(\omega_1,\ldots,\omega_n) \int_Y \beta^2 s(\omega_1,\ldots,\omega_n)$$

and equality holds if and only if α is proportional to β.

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- Idea: ∂_s perserves strictly Lorentzian polynomials
- Problem: if there are too many Kähler forms $\omega_1,\ldots,\omega_n$, the polynomial we start with might not be strictly Lorentzian.
- We use

  Lorentzian polynomials on cones (Brändén–Leaky ’23)

  to overcome this issue.