Entire Monge-Ampére equations and weighted Minkowski problems

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- $\alpha = n + 2$: Prescribed Gauss curvature as a function of the projection of points in the boundary,
- $\alpha = n + 1$: Aleksandrov problem.

Entire solutions

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We focus on entire solutions.

R-curvature

A widely studied problem (Aleksandrov, Pogorelov, Bakelman, Urbas...) is the Monge-Ampére equation for R-curvatures:

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Weak solutions:

$$\int_B f(x)\,dx = \int_{\partial u(B)} R(p)\,dp$$

for every Borel set $B \subset \mathrm{dom}(u)$, where

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By Caffarelli's regularity theory, under suitable assumptions weak solutions are classical solutions (e.g. Bielawski, '04).

Our generalization

We look for entire solutions to equations of the form

$$c\phi(Du(x), u^*(Du(x))) \det D^2u(x) = f(x),$$

where $u^*(p) = \sup_{x\in\mathbb{R}^n}\{x \cdot p - u(x)\}$, $c > 0$, $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}$.

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We can use a measure ρ instead of f:

$$\rho(B) = \omega(B, u, c\phi).$$

(Again, suitable regularity gives classical solutions)

Geometric idea

A continuous function $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ can be considered as the density of a measure μ on $\mathbb{R}^{n+1}$.

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We can consider the corresponding *weighted surface area measure* (Zvavitch, Lyvschitz, Fradelizi, Langharst, Kryvonos, Roysdon, Zhao...)

$$S_K^\mu(B) \coloneqq \int_{\tau_K(B)} \phi(X)\, d\mathcal{H}^n(X)$$

for every convex compact set K in $\mathbb{R}^{n+1}$, where τ_K is the *reverse spherical image*.

Weighted Minkowski problem

Theorem [Kryvonos and Langharst, '23]

Let μ be an even Borel measure on $\mathbb{R}^{n+1}$ satisfying

$$\lim_{r\to\infty} \frac{\mu(rB_{n+1}^2)^{\beta/n}}{r} = 0 \text{ and } \lim_{r\to 0^+} \frac{\mu(rB_{n+1}^2)^{\beta/n}}{r} = +\infty.$$

Suppose ρ is a finite, even Borel measure on $\mathbb{S}^n$ that is not concentrated in any great subsphere. Then, there exists a centrally symmetric convex compact set $K \subset \mathbb{R}^{n+1}$ such that

$$d\rho(\xi) = c_{\mu,K}\, dS_K^{\mu}(\xi), \quad c_{\mu,K} := \mu(K)^{\frac{\beta}{n}-1}.$$

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In the spirit of the geometric interpretations, what's the functional version of this problem?

Main result

Theorem [U., +'23]

Consider a Borel measure ρ on $\mathbb{R}^n$ that is not concentrated on an affine hyperplane. Consider, moreover, a continuous, and even function $\phi : \mathbb{R}^{n+1} \to [0, \infty)$. Then, if ρ has finite first moment, i.e.

$$\int_{\mathbb{R}^n} |x| \, d\rho(x) < +\infty,$$

and the measure μ with density ϕ with respect to the Lebesgue measure satisfies

$$\lim_{r\to\infty} \frac{\mu(rB^2_{n+1})^{\beta/n}}{r} = 0 \text{ and } \lim_{r\to 0^+} \frac{\mu(rB^2_{n+1})^{\beta/n}}{r} = +\infty,$$

there exist $c > 0$ and a convex function u such that for every Borel set $B \subset \mathbb{R}^n$

$$\omega(B, u, c\phi) = \rho(B).$$

Regularity

Using some tools from Caffarelli's regularity theory, we obtain the following.

Theorem [U., +'23]

In the hypotheses of the previous Theorem, suppose moreover that ρ has continuous density f with respect to the Lebesgue measure. If f and ϕ are such that there exists $c > 0$ such that $f, \phi > c$ and of class $C^{k,\alpha}$ for some $k \geqslant 0$ and $\alpha > 0$, then any weak solution is of class $C^{k+2,\alpha}$.

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  Equations explicitly depending on u usually (see e.g. Bakelman, require high regularity),
- Regularity extends directly to the weighted Minkowski problem,
- Lack of uniqueness: How can we prescribe an asymptotic cone?

Sketch of the proof

Step 1. Fix a vector $v \in \mathbb{S}^n$, $v^\perp \equiv \mathbb{R}^n$.

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$$\rho'(B) := \int_B \sqrt{1+|x|^2}\, d\rho(x).$$

We lift it as a measure on the sphere through the *inverse gnomonic projection*

$$L\colon \mathbb{R}^n \to \mathbb{S}^n_- = \{\xi \in \mathbb{S}^n \colon \xi \cdot v < 0\}$$
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We extend the measure by symmetry (here hides the non-uniqueness)$\rightarrow$ by the weighted version of the Minkowski problem, exists K!

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To sum up:

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Core idea: $x = L^{-1} \circ \tau_K^{-1} \circ \pi^{-1} \circ \partial u(x)$

Step 3. We have the corresponding changes of variables:

$$\begin{aligned}\omega(B,u,c\phi) &= \int_{\partial u(B)} c\phi((x,w(x)))\,dx = \int_{\pi^{-1}\circ\partial u(B)} \frac{c\phi(y)}{\sqrt{1+|Dw(\pi(y))|^2}}\,d\mathcal{H}^n(y) \\ &= \int_{\tau_K^{-1}\circ\pi^{-1}\circ\partial u(B)} |\xi\cdot v|\,c_{\mu,K}dS_K^{\mu}(\xi) = \int_{L^{-1}\circ\tau_K^{-1}\circ\pi^{-1}\circ\partial u(B)} \frac{1}{\sqrt{1+|z|^2}}d\rho'(z) \\ &= \int_{L^{-1}\circ\tau_K^{-1}\circ\pi^{-1}\circ\partial u(B)} d\rho(z) = \rho(B).\end{aligned}$$

THANKS FOR YOUR ATTENTION!