The Uniqueness of the Gauss Image Measure

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$\mathcal{K}_o^n$ is the set of convex bodies with the center at their interior.
∂K is the boundary of K.
The radial map $r_K : S^{n-1} \to \partial K$ is defined by

$$r_K(u) = ru \in \partial K. \tag{1}$$

By $N(K, x)$, we denote the *normal cone of K at $x \in \partial K$*, that is the set of all outer unit normals at x:

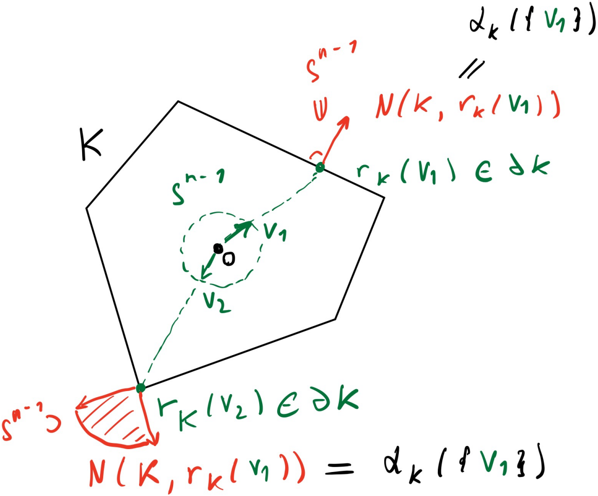
$$N(K, x) = \{v \in S^{n-1} : (y - x) \cdot v \leq 0 \text{ for all } y \in K\}. \tag{2}$$

We define the radial Gauss image of $\omega \subset S^{n-1}$ as:

$$\boldsymbol{\alpha}_K(\omega) = \bigcup_{x \in r_K(\omega)} N(K, x) \subset S^{n-1}. \tag{3}$$

The radial Gauss image $\boldsymbol{\alpha}_K$ maps sets of S^{n-1} into sets of S^{n-1}.

The radial Gauss Image Map, $\alpha_K(\cdot)$ is a set valued map, which is a composition of radial map r_K and the multivalued Gauss Map.



Definition (K. J. Böröczky, E. Lutwak, D. Yang, G. Y. Zhang and Y. M. Zhao, 2019)

The Gauss image measure of λ *via* K, is a measure defined as the pushforward of the λ via map $\boldsymbol{\alpha}_K$. That is for each borel $\omega \subset S^{n-1}$

$$\lambda(\boldsymbol{\alpha}_K(\omega)) = \lambda(K, \omega) \tag{4}$$

1. λ is spherical Lebesgue measure $\implies \lambda(K, \cdot)$ is Alexandrov's integral curvature
2. λ is Federer's $(n-1)^{\text{th}}$ curvature measure $\implies \lambda(K, \cdot)$ is the surface area measure of Alexandrov-Fenchel-Jessen
3. Dual curvature measures (the dual counterparts of Federer's curvature measures) are also Gauss Image Measures

Question

Given that $\lambda(K,\cdot) = \lambda(L,\cdot)$ what can we say about K and L?

- λ is spherical Lebesgue measure $\implies K = cL$ for some $c > 0$ (Aleksandrov)
- λ is absolutely continuous and $\text{spt}\lambda = S^{n-1} \implies K = cL$ for some $c > 0$ (BLYZZ, 2019) (Also Bertrand, from mass transport point of view. Cost function: $-\log(u,v)$.)

$$\text{spt}\lambda = \{v \in S^{n-1} \mid \text{for every open neighborhood } N_v \text{ of v, } \lambda(N_v) > 0\}$$

Theorem (From measures to maps, S. 2023)

Suppose $\lambda(K,\cdot), \lambda(L,\cdot)$ are finite Borel measures for Borel measure λ. Then,

$$\lambda(K,\cdot) = \lambda(L,\cdot)$$

if and only if $\boldsymbol{\alpha}_{K^*} = \boldsymbol{\alpha}_{L^*}$ ***almost everywhere as multivalued maps.***

The immediate guess for the definition would be that:

$$\lambda(\{v \mid \boldsymbol{\alpha}_{K^*}(v) \neq \boldsymbol{\alpha}_{L^*}(v)\}) = 0 \tag{5}$$

Yet, this is not the good definition for this class of maps.

Consider two measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ and some measure λ on the domain, then

$$\begin{gathered} \lambda(\{x \mid f(x) \neq g(x)\}) = 0 \\ \Leftrightarrow \\ \forall \omega \text{ Borel } \lambda(f^{-1}(\omega) \triangle g^{-1}(\omega)) = 0, \end{gathered} \tag{6}$$

where by triangle we denote the symmetric difference of the sets.

$$A \triangle B = (A \setminus B) \cup (B \setminus A) \tag{7}$$

Now consider two set-valued functions (similar to that of radial Gauss Image map behaviour) f, g from subsets of $\mathbb{R}$ to subsets of $\mathbb{R}$

$$\begin{aligned} f(\omega) &= \bigcup_{x\in\omega}(x-1, x+1) \\ g(x) &= \bigcup_{x\in\omega}[x-1, x+1] \end{aligned} \tag{8}$$

For these two functions $f(x) \neq g(x)$ at every point x, but yet

$$\forall\omega \text{ Borel } \lambda(f^{-1}(\omega)\triangle g^{-1}(\omega)) = 0, \tag{9}$$

Note that for set valued function we define

$$f^{-1}(\omega) = \{x \mid f(x) \cap \omega \neq \varnothing\} \tag{10}$$

In other words, we care about two set-valued functions *mapping to roughly the same sets for a given point* rather than *mapping exactly the same for almost everywhere point*.

Definition (S. 2023)

Two set valued functions are equal almost everywhere with respect to measure λ if for any ω Borel:

$$\lambda(f^{-1}(\omega)\triangle g^{-1}(\omega)) = 0 \tag{11}$$

Alternatively, one can think about this in terms of symmetric difference pseudo metric space (The Nikodym Metric Space)

Class of functions

The large class of set-valued functions are subdifferential (subderivative) of convex functions!

In fact, one can view the inverse Gauss Image maps of bodies K and L as the gradient of the support functions of convex bodies K and L.

Theorem (From measures to maps, S. 2023)

Suppose $\lambda(K,\cdot), \lambda(L,\cdot)$ are finite Borel measures for Borel measure λ. Then,

$$\lambda(K,\cdot) = \lambda(L,\cdot)$$

*if and only if $\boldsymbol{\alpha}_{K^*} = \boldsymbol{\alpha}_{L^*}$ almost everywhere as **multivalued maps**, that is*

$$\forall \omega \subset S^{n-1} \text{ Borel sets } \lambda(\boldsymbol{\alpha}_K(\omega) \triangle \boldsymbol{\alpha}_L(\omega)) = 0. \tag{12}$$

Is the same as saying

$$\begin{gathered} \forall \omega \text{ Borel sets } \lambda(\boldsymbol{\alpha}_K(\omega)) = \lambda(\boldsymbol{\alpha}_L(\omega)). \\ \Leftrightarrow \\ \forall \omega \text{ Borel sets } \boldsymbol{\alpha}_K(\omega) = \boldsymbol{\alpha}_L(\omega) \text{ up to a } \lambda \text{ measure zero set.} \end{gathered} \tag{13}$$

This is quite special behavior of radial Gauss Image maps. For example, take λ to be uniform measure and rotations of the sphere instead of $\boldsymbol{\alpha}_K$ and $\boldsymbol{\alpha}_L$.

Theorem (From measure theory to continuity, S. 2023)

Let $K, L \in \mathcal{K}_o^n$. Suppose $\lambda(K,\cdot) = \lambda(L,\cdot)$ are finite Borel measures for a spherical submeasure λ. Then, given $u \in spt\lambda$,

$$\boldsymbol{\alpha}_{K^*,L^*}(u) := \boldsymbol{\alpha}_{K^*}(u) \cap \boldsymbol{\alpha}_{L^*}(u) \neq \varnothing \tag{14}$$

In particular, $\boldsymbol{\alpha}_{K^*,L^*}$ defined on $spt\lambda$ is a continuous map. That is, for any $\varepsilon > 0$ there exist $\delta > 0$ such that for any $u \in spt\lambda$

$$\boldsymbol{\alpha}_{K^*,L^*}(u_\delta) \subset \boldsymbol{\alpha}_{K^*,L^*}(u)_\varepsilon. \tag{15}$$

where for $\omega \subset S^{n-1}$ we define its *outer parallel set* ω_α to be

$$\omega_\alpha = \bigcup_{u \in \omega} \{v \in S^{n-1} : u \cdot v > \cos\alpha\}. \tag{16}$$

Given $f, g : \mathbb{R}^2 \to \mathbb{R}$, let $\gamma(t) : [0, 1] \to \mathbb{R}^2$ be a path of finite length. Let ∂f and ∂g be subdifferentials for f and g (on $\mathbb{R}^2$). Then,

$$\begin{gathered} \text{If } \forall t \text{ we have } \partial f(\gamma(t)) \cap \partial g(\gamma(t)) \neq \varnothing \\ \Rightarrow \\ f(\gamma(t)) = g(\gamma(t)) + c \end{gathered} \tag{17}$$

$$\boldsymbol{\alpha}_{K^*,L^*}(u) := \boldsymbol{\alpha}_{K^*}(u) \cap \boldsymbol{\alpha}_{L^*}(u) \neq \varnothing \text{ for } u \in \operatorname{spt}\lambda \tag{18}$$

$$\boldsymbol{\alpha}_{K^*,L^*}(u_\delta) \subset \boldsymbol{\alpha}_{K^*,L^*}(u)_\varepsilon \text{ for } u \in \operatorname{spt}\lambda \tag{19}$$

We obtain the following:

Theorem (From α_K and α_L to K and L, S. 2023)

Let $K, L \in \mathcal{K}_o^n$. Suppose $\lambda(K,\cdot) = \lambda(L,\cdot)$ are finite Borel measures for a spherical measure λ, defined on the Lebesgue measurable subsets of S^{n-1}. Then on each rectifiable path connected component $D \subset \operatorname{spt}\lambda$, K^* and L^* are are equal up to a dilation. Alternatively, for each $v_1, v_2 \in D$ we have

$$\frac{h_K(v_1)}{h_L(v_1)} = \frac{h_K(v_2)}{h_L(v_2)}, \tag{20}$$

where by h_K and h_L we denote the support functions of K and L.

In particular, one can think about this in terms of tangential bodies.

Given C^1 function $f, g : [0,1] \to \mathbb{R}$

$$f' = g' \Rightarrow f = g + c \tag{21}$$

This is MVT. Yet, this doesn't generalize to higher dimensions, in naive approach:

In 1935, Whitney constructed a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla f = 0$ on a curve $D \subset \mathbb{R}^2$ such that f is not constant on D.

The caviat being, that the curve is a fractal, and has an infinite length.

Question

Can one construct two convex functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ and some curve $D \subset \mathbb{R}^2$ such that $\partial f = \partial g$ on this curve and, yet, $f \neq g + c$?

Theorem (From α_K and α_L to K and L, S. 2023)

Let $K, L \in \mathcal{K}_o^n$. Suppose $\lambda(K, \cdot) = \lambda(L, \cdot)$ are finite Borel measures for a spherical measure λ, defined on the Lebesgue measurable subsets of S^{n-1}. Then on each rectifiable path connected component $D \subset \mathrm{spt}\lambda$, K^* and L^* are are equal up to a dilation. Alternatively, for each $v_1, v_2 \in D$ we have

$$\frac{h_K(v_1)}{h_L(v_1)} = \frac{h_K(v_2)}{h_L(v_2)}, \tag{22}$$

where by h_K and h_L we denote the support functions of K and L.

Now, in the remaining time we will address the most crucial part of the proof of the first statement:

Theorem (From measures to maps, S. 2023)

Suppose $\lambda(K,\cdot), \lambda(L,\cdot)$ are finite Borel measures for Borel measure λ. Then,

$$\lambda(K,\cdot) = \lambda(L,\cdot)$$

*if and only if $\boldsymbol{\alpha}_{K^*} = \boldsymbol{\alpha}_{L^*}$ almost everywhere as **multivalued maps**, that is*

$$\forall \omega \subset S^{n-1} \text{ Borel sets } \lambda(\boldsymbol{\alpha}_K(\omega) \triangle \boldsymbol{\alpha}_L(\omega)) = 0. \tag{23}$$

Definition

Given $t \in [0,1]$ we define the harmonic mean of $K, L \in \mathcal{K}_o^n$ as

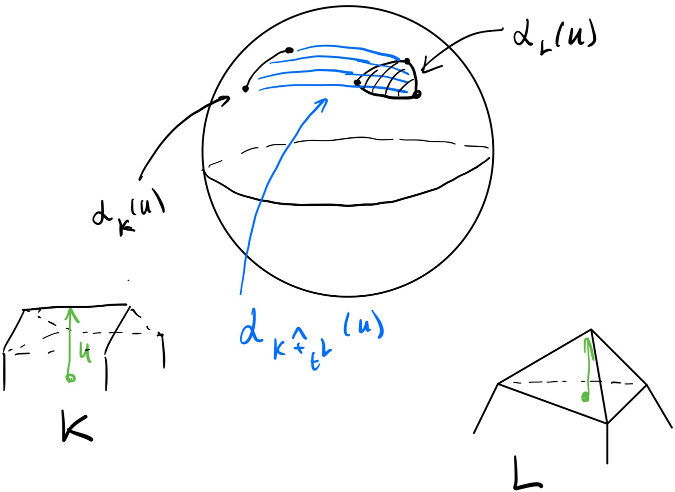
$$K \hat{+}_t L := ((1-t)K^* + tL^*)^*. \tag{24}$$

Using this, the essential ingredient in the proof of main Theorem is to show that

$$\boldsymbol{\alpha}_K(\gamma) \triangle \boldsymbol{\alpha}_L(\gamma) \setminus \left(\boldsymbol{\alpha}_K(\partial\gamma) \cup \boldsymbol{\alpha}_L(\partial\gamma)\right) \subset \bigcup_{0<t<1} \boldsymbol{\alpha}_{K \hat{+}_t L}(\partial\gamma)$$

Proposition (S. 2023)

Given $u \in S^{n-1}$, $\boldsymbol{\alpha}_{K \hat{+}_t L}(u)$ is a variation from $\boldsymbol{\alpha}_K(u)$ to $\boldsymbol{\alpha}_L(u)$ along geodesic segments on S^{n-1}.



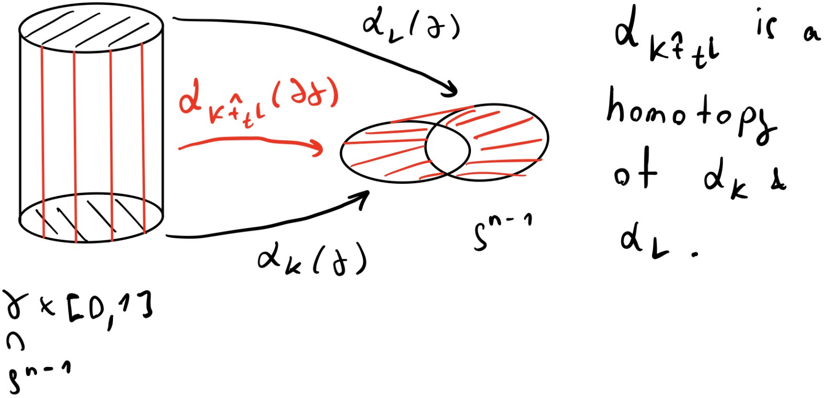
When K, L are C^1 strictly convex bodies $\boldsymbol{\alpha}_K, \boldsymbol{\alpha}_L$ is a homeomorphism of S^{n-1} to S^{n-1}. Then, for the previous equation it is sufficient to establish:

$$\boldsymbol{\alpha}_K(\gamma) \triangle \boldsymbol{\alpha}_L(\gamma) \subset \bigcup_{0 \leq t \leq 1} \boldsymbol{\alpha}_{K \hat{+}_t L}(\partial\gamma). \tag{25}$$

In this case, notice that $\boldsymbol{\alpha}_{K \hat{+}_{\cdot} L} : \gamma \times [0,1] \to S^{n-1}$ defines a homotopy of homeomorphisms $\boldsymbol{\alpha}_K$ and $\boldsymbol{\alpha}_L$.

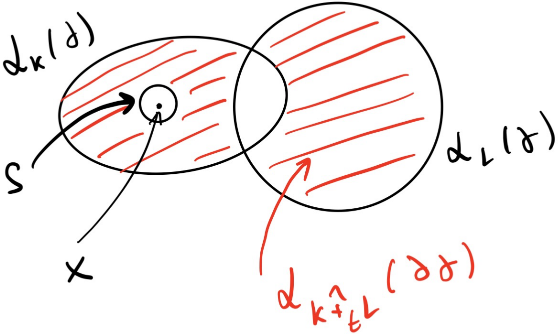
$$\boldsymbol{\alpha}_K(\gamma)\triangle\boldsymbol{\alpha}_L(\gamma) \subset \bigcup_{0\le t\le 1} \boldsymbol{\alpha}_{K\hat{+}_t L}(\partial\gamma). \tag{26}$$

In C^1 strictly convex case, (27) reduces to the following geometric picture:

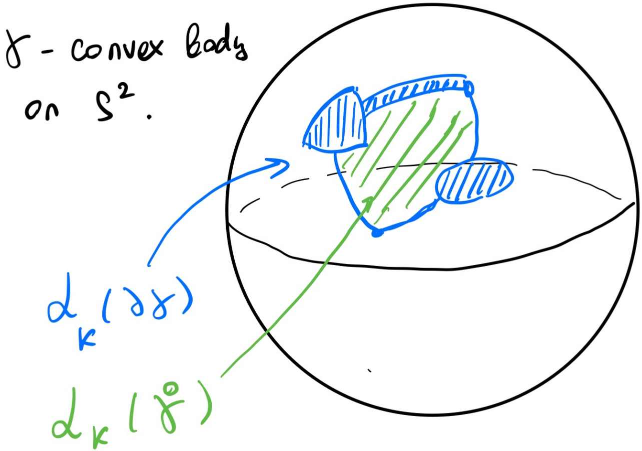


$$x \in \bigcup_{0 \le t \le 1} \boldsymbol{\alpha}_{K \hat{+}_t L}(\partial\gamma) \setminus \boldsymbol{\alpha}_K(\gamma) \triangle \boldsymbol{\alpha}_L(\gamma). \tag{27}$$

If there exist such x, we can define a projection P onto sphere S centered at point x. Then $P \circ \boldsymbol{\alpha}_K(\partial\gamma)$ covers the sphere (degree of $P \circ \boldsymbol{\alpha}_K(\partial\gamma)$ is ≥ 1), yet degree of $P \circ \boldsymbol{\alpha}_L(\partial\gamma)$ is zero, so $\alpha_K(\partial\gamma)$ is not homotopic to $\alpha_L(\partial\gamma)$ by Hopf Theorem.



But in general, map α_K is more difficult:



Proposition (S. 2023)

Given any $\omega \subset S^{n-1}$, $\boldsymbol{\alpha}_{K\hat{+}_t L}(\omega) : [0,1] \to S^{n-1}$ *is Lipschitz continuous map from t to sets on sphere equipped with Hausdorff distance* d_H*:*

$$d_H(\boldsymbol{\alpha}_{K\hat{+}_{t_1} L}(\omega), \boldsymbol{\alpha}_{K\hat{+}_{t_2} L}(\omega)) \leq 2\max(\frac{R_K}{r_K}, \frac{R_L}{r_L})\max(\frac{R_K}{r_L}, \frac{R_L}{r_K})|t_1 - t_2|. \quad (28)$$

Using this Proposition, using continuity arguments we manage to show that the same result holds:

$$\boldsymbol{\alpha}_K(\gamma)\triangle\boldsymbol{\alpha}_L(\gamma) \setminus \left(\boldsymbol{\alpha}_K(\partial\gamma) \cup \boldsymbol{\alpha}_L(\partial\gamma)\right) \subset \bigcup_{0<t<1} \boldsymbol{\alpha}_{K\hat{+}_t L}(\partial\gamma) \quad (29)$$

Thank you!