# Expected valuations of random zonotopes

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Let $X_1, \dots, X_n$ be stochastically independent, identically distributed random points in $\mathbb{R}^d$.

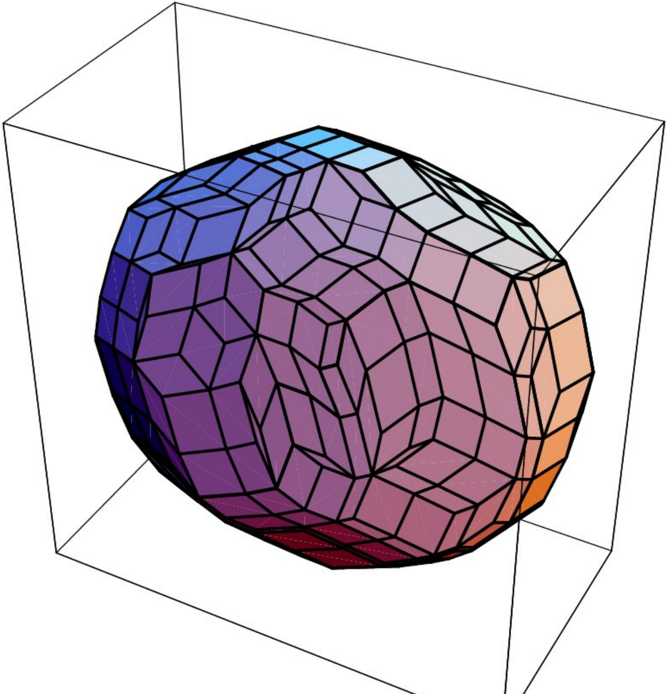
There is a huge literature about the convex hull of $X_1, \dots, X_n$.

Instead, we propose to consider the Minkowski sum of the segments $\bar{X}_1, \dots, \bar{X}_n$, where

$$\bar{x} := [o, x] := \mathrm{conv}\{o, x\} \quad \text{for } x \in \mathbb{R}^d.$$

The sum $\bar{X}_1 + \cdots + \bar{X}_n$ is a random zonotope.

A picture of a zonotope:



Our starting point is a result of Richard A. Vitale (1991):

**Theorem.** *Let $X$ be a random vector in $\mathbb{R}^d$ with $\mathbb{E}\|X\| < \infty$. Let $M_X$ be a $d \times d$ matrix whose columns are i.i.d. copies of $X$. Then*

$$\mathbb{E}|\det M_X| = d!\, V_d(\mathsf{Z}_X),$$

*where $\mathsf{Z}_X$ is the selection expectation of $\bar{X}$.*
*($V_d$ = volume in $\mathbb{R}^d$).*

**Explanation**
(e.g., Molchanov, Theory of Random Sets (2005))

The *selection expectation* of an integrably bounded random closed set is the closure of the set of all expectations of integrable selections of the set.

Fortunately, in our case, $Z_X$ is a convex body, and the support functions satisfy

$$h(Z_X, u) = \mathbb{E}h(\bar{X}, u) = \int_{\mathbb{R}^d} h(\bar{x}, u)\, \mathbb{P}_X(\mathrm{d}x) \quad \text{for } u \in \mathbb{R}^d,$$

where $\mathbb{P}_X$ is the distribution of $X$.

Thus, $Z_X$ can be approximated by finite sums of segments and hence is a zonoid.

Vitale's result can be interpreted geometrically:

Since the absolute determinant of a quadratic matrix is the volume of a parallelepiped, we have

$$\mathbb{E} V_d(\bar{X}_1 + \cdots + \bar{X}_d) = d!\, V_d(\mathrm{Z}_X),$$

if $X_1, \ldots, X_d$ are i.i.d. copies of $X$.

This calls for generalizations.

(1) Can it be extended to more than $d$ summands? Yes.

(2) Can the volume be replaced by an intrinsic volume? Yes.

(3) Can the intrinsic volume $V(K[j], B^d[d-j])$ be replaced by a mixed volume $V(K[j], C_1, \ldots, C_{d-j})$ (with fixed $C_1, \ldots, C_{d-j}$)?

Also here, the answer is Yes.

But now, a theorem of Alesker (2001), proving a conjecture of McMullen (1980), comes to mind:

The functionals $K \mapsto V(K[j], C_1, \ldots, C_{d-j})$, $C_1, \ldots, C_{d-j} \in \mathcal{K}^d$, are dense in $\mathbf{Val}_j$, the space of translation invariant, continuous, $j$-homogeneous valuations on the convex bodies in $\mathbb{R}^d$.

Can this be used to extend the result to $\mathbf{Val}_j$?

Yes, but fortunately a more elementary approach is possible.

We have the following result:

**Theorem.** *Let $X$ be a random vector in $\mathbb{R}^d$ with $\mathbb{E}\|X\| < \infty$. Use its distribution $\mathbb{P}_X$ to define a deterministic zonoid $\mathsf{Z}_X$ with support function*

$$h(\mathsf{Z}_X, \cdot) = \int_{\mathbb{R}^d} h(\bar{x}, \cdot)\, \mathbb{P}_X(\mathrm{d}x).$$

*Let $X_1, \ldots, X_n$, with $n \geq j \in \{1, \ldots, d\}$, be i.i.d. copies of $X$, and define the random zonotope*

$$Z_n := \frac{1}{n}(\bar{X}_1 + \cdots + \bar{X}_n).$$

*If $\varphi \in \mathbf{Val}_j$, then*

$$\mathbb{E}\varphi(Z_n) = \frac{n!}{n^j (n-j)!} \varphi(\mathsf{Z}_X).$$

# The essential steps of the proof

**(1)** A polynomiality result of McMullen (1974):

There exists a symmetric mapping $\Phi : (\mathcal{K}^d)^j \to \mathbb{R}$, continuous, translation invariant, Minkowski additive in each variable, such that

$$\varphi(\lambda_1 K_1 + \cdots + \lambda_n K_n)$$

$$= \sum_{r_1,\ldots,r_n=0}^{j} \binom{j}{r_1 \ldots r_n} \lambda_1^{r_1} \cdots \lambda_n^{r_n} \Phi(K_1[r_1], \ldots, K_n[r_n]).$$

**(2)** The fact that $\varphi(K) = 0$ if $\dim K < j$ leads to a simplification for segments, namely

$$\varphi(\bar{x}_1 + \cdots + \bar{x}_n) = j! \sum_{1 \le i_1 < \cdots < i_j \le n} \Phi(\bar{x}_{i_1}, \ldots, \bar{x}_{i_j}).$$

**(3)** Let $j \leq n \leq k$ (think of large $k$). Then (2) leads to

$$\varphi(\bar{x}_1 + \cdots + \bar{x}_k) = \binom{k-j}{n-j}^{-1} \sum_{1 \leq i_1 < \cdots < i_n \leq k} \varphi(\bar{x}_{i_1} + \cdots + \bar{x}_{i_n}).$$

**(4)** With $Z_k := \frac{1}{k}(\bar{X}_1 + \cdots + \bar{X}_k)$ we get

$$\varphi(Z_k) = \frac{1}{k^j} \binom{k-j}{n-j}^{-1} \binom{k}{n} U_k^{(n)}(h)$$

with the *U*-statistic

$$U_k^{(n)}(h) := \binom{k}{n}^{-1} \sum_{1 \leq i_1 < \cdots < i_n \leq k} h(X_{i_1}, \ldots, X_{i_n})$$

of order $n$ with kernel function

$$h(x_1, \ldots, x_n) := \varphi(\bar{x}_1 + \cdots + \bar{x}_n), \quad x_1, \ldots, x_n \in \mathbb{R}^d.$$

**(5)** The strong law for $U$-statistics by Hoeffding (1961) says that

$$\lim_{k\to\infty} U_k^{(n)}(h) = \mathbb{E}h(X_1,\dots,X_n) \quad \text{almost surely,}$$

hence

$$\lim_{k\to\infty} \varphi(Z_k) = \frac{(n-j)!}{n!} n^j \mathbb{E}\varphi(Z_n) \quad \text{a.s.}$$

**(6)** The strong law for random sets by Artstein and Vitale (1975) says that

$$\lim_{k\to\infty} Z_k = \mathbb{E}\bar{X} = Z_X \quad \text{a.s.}$$

in the Hausdorff metric, hence (using that $\varphi$ is continuous)

$$\lim_{k\to\infty} \varphi(Z_k) = \varphi(Z_X) \quad \text{a.s.}$$

**(7)** From (5) and (6) together,

$$\frac{(n-j)!}{n!} n^j \mathbb{E}\varphi(Z_n) = \lim_{k\to\infty} \varphi(Z_k) = \varphi(Z_X) \Rightarrow \text{Assertion.}$$

# A central limit theorem

Recall the $U$-statistic of order $j$ with kernel $h$, for a random sample $(X_1, \dots, X_n)$ of size $n \geq j$,

$$U_n^{(j)}(h) = \binom{n}{j}^{-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n} h(X_{i_1}, \dots, X_{i_j}).$$

There is a central limit theorem for $U$-statistics, by Hoeffding (1948). It requires two conditions:

**(a)** $\mathbb{E}h^2(X_1, \dots, X_j) < \infty$,
**(b)** $\zeta_1 > 0$, where

$$\zeta_1 := \mathbb{E}\widetilde{h}_1^2(X), \qquad \widetilde{h}_1 := h_1 - \theta,$$

$$h_1(x) := \mathbb{E}h(x, X_2, \dots, X_j), \qquad \theta := \mathbb{E}h(X_1, \dots, X_j).$$

Under these assumptions, the central limit theorem says that, as $n \to \infty$,

$$\sqrt{n}\left(U_n^{(j)}(h) - \theta\right) \xrightarrow{d} \mathcal{N}(0, j^2\zeta_1),$$

where $\mathcal{N}(0, j^2\zeta_1)$ is a normally distributed random variable with expectation 0 and variance $j^2\zeta_1$.

Let's see how the assumptions (a) and (b) can be satisfied in our special case.

Here we have

$$h(x_1, \dots, x_n) = \varphi(\bar{x}_1 + \dots + \bar{x}_n), \quad x_1, \dots, x_n \in \mathbb{R}^d.$$

**(a)** We need $\mathbb{E}h^2(X_1, \ldots, X_j) < \infty$.

Write $\bar{X} = \|X\|s$, where $s$ is a random segment with endpoints $o$ and a unit vector.
For $n = j$, the previous simplification gives

$$\varphi(\bar{x}_1 + \cdots + \bar{x}_j) = j!\Phi(\bar{x}_1, \ldots, \bar{x}_j),$$

hence

$$h(X_1, \ldots,_j) = j!\Phi(\bar{X}_1, \ldots, \bar{X}_j) = j!\|X_1\| \cdots \|X_j\|\Phi(s_1, \ldots, s_j),$$

Here it was used that $\Phi$ is Minkowski linear in each variable. It follows that

$$\mathbb{E}h^2(X_1, \ldots, X_j) = (j!)^2\mathbb{E}[\|X_1\| \cdots \|X_j\|\Phi(s_1, \ldots, s_j)]^2 < \infty$$

if $\mathbb{E}\|X\|^2 < \infty$, since $\Phi$ is continuous and hence attains a maximum on a compact set of convex bodies.

**(b)** We need $\zeta_1 > 0$.

First, we have

$$\theta = \mathbb{E}h(X_1, \ldots, X_j) = \mathbb{E}\varphi(\bar{X}_1 + \cdots + \bar{X}_j) = j^j \mathbb{E}\varphi(Z_j) = \varphi(Z_X),$$

by our first theorem.

Second, for $x \in \mathbb{R}^d$ we have

$$h_1(x) = \mathbb{E}h(x, \bar{X}_2 \ldots, \bar{X}_j) = \mathbb{E}\varphi(\bar{x} + \bar{X}_2 + \cdots + \bar{X}_j).$$

We define a random zonotope $Z_n(x)$ by

$$Z_n(x) := \bar{x} + \frac{1}{n}(\bar{X}_2 + \cdots + \bar{X}_n) \stackrel{d}{=} \bar{x} + \frac{n-1}{n} Z_{n-1}.$$

Then, with probability one (by Artstein–Vitale) $Z_n(x) \to x + Z_X$ as $\to \infty$.

Thus,

$$\lim_{n\to\infty} \varphi(Z_n(x)) = \varphi(x + Z_X) \quad \text{a.s.}$$

From a previous identity, together with properties of $\Phi$, we obtain

$$\varphi(Z_n(x)) = \frac{1}{n^{j-1}} \binom{n-1}{j-1} U_{n-1}^{(j-1)}(g_x) + \frac{1}{n^j} U_{n-1}^{(j)}(h)$$

with $g_x(x_2, \ldots, x_j) := \varphi(\bar{x} + \bar{x}_2 + \cdots + \bar{x}_j)$.

The strong law for U-statistics gives that, with probability one,

$$\lim_{n\to\infty} \varphi(Z_n(x)) = \frac{1}{(j-1)!} \mathbb{E}\varphi(\bar{x}_2 + \cdots + \bar{x}_j) + \frac{1}{j!} \mathbb{E}\varphi(\bar{x}_1 + \cdots + \bar{x}_j).$$

Both limit theorems together give

$$h_1(x) = (j-1)![\varphi(\overline{x} + Z_X) - \varphi(Z_X)].$$

Recall that

$$\zeta_1 = \mathbb{E}[h_1(X) - \theta]^2.$$

Hence, to achieve that $\zeta_1 > 0$, we need assumptions to ensure that **NOT**

$$\varphi(x + Z_X) = (j+1)\varphi(Z_X) \quad \text{for all } x \in \operatorname{supp} \mathbb{P}_X.$$

Hence, we assume the following:

(1) $\mathbb{E}\|X\|^2 < \infty$,

(2) The support of $\mathbb{P}_X$ contains $o$ and is not contained in some $(j-1)$-dimensional linear subspace,

(3) $\varphi(K) \neq 0$ if $\dim K \geq j$.

**Theorem.** *Under these assumptions, as $n \to \infty$,*

$$\sqrt{n}\,(\varphi(Z_n) - \varphi(\mathrm{Z}_X)) \xrightarrow{d} \mathcal{N}(0, (j!j)^2\zeta_1).$$

**Thank you for your attention!**