On a j-Santaló Conjecture

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Polarity

- Let K be a symmetric (i.e. $K = -K$) convex body (i.e. compact convex with non-empty interior in $\mathbb{R}^n$).
- The polar body of K is the convex body given

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A trivial generalization

Recall

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1\}.$$

Thus, if K_1, K_2 are symmetric sets (not necessarily convex) with

$$\langle x, y \rangle \leq 1, \forall x \in K_1, \forall y \in K_2,$$

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A functional version

- Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be a function. Its polar is defined as

$$\begin{aligned} f^\circ(x) : &= \inf_{y \in \mathbb{R}^n} (e^{-\langle x,y \rangle}/f(y)) \\ &= e^{-\mathcal{L}(-\log f)}(x), \end{aligned}$$

  where $\mathcal{L}$ denotes the Legendre transform.

- Theorem (Ball '86, Artstein-Klartag-Milman '05, Lehec '09): If f is even, then

$$\int_{\mathbb{R}^n} f(x)dx \int_{R^n} f^\circ(x)dx \leq \int_{\mathbb{R}^n} e^{-|x|^2/2}dx \int_{R^n} (e^{-|x|^2/2})^\circ dx$$

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- Again, if f_1, f_2 (both even) satisfy $f_1(x_1)f_2(x_2) \leq e^{-\langle x_1, x_2 \rangle}$, for all $(x_1, x_2) \in K_1 \times K_2$, then

$$\int_{\mathbb{R}^n} f_1 \int_{\mathbb{R}^n} f_2 \leq \left(\int_{\mathbb{R}^n} e^{-|x|^2/2} dx \right)^2 .$$

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Functional Santaló for many functions (?)

Conjecture

(Kolesnikov-Werner '20) Let $k \geq 2$ be an integer, $\rho : \mathbb{R} \to \mathbb{R}_+$ be a decreasing function and $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}_+$ be even integrable functions, such that

$$\prod_{i=1}^{k} f_i(x_i) \leq \rho\left(\sum_{1 \leq i < l \leq k} \langle x_i, x_l \rangle \right), \qquad \forall x_1, \ldots, x_k \in \mathbb{R}^n. \tag{1}$$

Then, it holds

$$\prod_{i=1}^{k} \int_{\mathbb{R}^n} f_i(x_i)\, dx_i \leq \left(\int_{\mathbb{R}^n} \rho\left(\frac{k(k-1)}{2} \|u\|_2^2 \right)^{1/k} du \right)^k .$$

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What is polarity for many sets or functions?

- Who knows?
- Recall polarity condition: For $k = 2$, $\langle x_1, x_2 \rangle \leq 1$, for all $x_1 \in K_1$, $x_2 \in K_2$ or $f_1(x_1)f_2(x_2) \leq \rho(\langle x_1, x_2 \rangle)$, for all $x_1, x_2 \in \mathbb{R}^n$.

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j-polarity condition

- For $j \in \{1, \dots, k\}$ and for reals $r_1, \dots, r_k$, set

$$s_j(r_1, \dots, r_k) := \sum_{1 \le i_1 < \dots < i_j \le k} r_{i_1} \cdots r_{i_j}.$$

- For $x_1, \dots, x_k \in \mathbb{R}^n$, with $x_i = (x_i(1), \dots, x_i(n))$, set

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and

$$\mathcal{E}_j := \frac{\mathcal{S}_j}{\binom{k}{j}}.$$

- $S_2(x_1, \dots, x_k) = \sum_{1 \le i < l \le k} \langle x_i, x_l \rangle.$

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- We say that the sets $K_1, \dots, K_k$ satisfy $\mathcal{E}_j$-polarity condition if for all $x_1 \in K_1, \dots, x_k \in K_k$, it holds

$$\mathcal{E}_j(x_1, \dots, x_k) \leq 1.$$

- We say that the functions $f_1, \dots, f_k$ satisfy $\mathcal{S}_j$-polarity condition with respect to a decreasing function ρ, if for all $x_1, \dots, x_k \in \mathbb{R}^n$, it holds

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j-Santaló conjectures

Conjecture

(j-Santaló conjecture) Let $2 \leq j \leq k$, where $k \geq 2$. If $K_1, \dots, K_k$ are symmetric convex bodies, satisfying $\mathcal{E}_j$-polarity condition, then

$$\prod_{i=1}^{k} |K_i| \leq |B_j^n|^k. \tag{2}$$

Conjecture

(Functional j-Santaló conjecture) Let $2 \leq j \leq k$, where $k \geq 2$. If $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}_+$ are even integrable functions, satisfying $\mathcal{S}_j$-polarity condition with respect to some decreasing function $\rho : \mathbb{R} \to [0, \infty]$, then

$$\prod_{i=1}^{k} \int_{\mathbb{R}^n} f_i(x_i)\, dx_i \leq \left(\int_{\mathbb{R}^n} \rho\left(\binom{k}{j} \|u\|_j^j \right)^{1/k} du \right)^k. \tag{3}$$

Remarks

- If $j = 2$, then the Functional j-Santaló conjecture is just the Kolesnikov-Werner conjecture.
- Functional j-Santaló $\Rightarrow$ j-Santaló. Indeed, take $f_i := 1_{K_i}, \ i = 1, \dots, k$

$$\text{and} \qquad \rho(t) := \begin{cases} +\infty, & t < 0 \\ 1_{[0,1]}\left(\binom{k}{j}^{-1} t\right) & t \geq 0 \end{cases}.$$

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- We exclude the case $j = 1$, because the quantity $|K_1| \dots |K_k|$ can be unbounded for bodies $K_1, \dots, K_k$ satisfying $\mathcal{E}_1$-polarity condition. This can be seen by taking all K_i to be the symmetric slab $\{x \in \mathbb{R}^n : |x_1 + \dots + x_n| \leq 1\}$.

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Main (partial) results

Theorem

The j-Santaló Conjecture holds in the following cases:

1. $K_1, \dots, K_k$ *are unconditional convex bodies.*
2. $j = k$.
3. j *is even and* $K_3, \dots, K_k$ *are unconditional convex bodies.*

Moreover, in all three cases, (2) is sharp for
$K_1 = K_2 = \dots = K_k = B_j^n$.

Theorem

The functional j-Santaló Conjecture holds in the following cases:

1. $f_1, \dots, f_k$ *are unconditional functions.*
2. $j = k$.
3. j *is even and* $f_3, \dots, f_k$ *are unconditional functions.*

Idea of Proof

- Work with bodies instead of functions.
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The unconditional case

Theorem(1-dimensional multiplicative Prékopa-Leindler inequality). If some integrable functions $h, h_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, k$, satisfy

$$\prod_{i=1}^{k} h_i(t_i)^{\frac{1}{k}} \leq h\left(\prod_{i=1}^{k} t_i^{\frac{1}{k}}\right), \qquad \forall t_i > 0,\ i = 1\ldots, k,$$

then it holds

$$\prod_{i=1}^{k}\left(\int_{\mathbb{R}_+} h_i(t_i)\, dt_i\right)^{\frac{1}{k}} \leq \int_{\mathbb{R}_+} h(t)\, dt.$$

The unconditional case

- Use Keith Ball's inductive argument and the PL inequality to obtain
  Proposition. Let $1 \leq j \leq k$ be two integers, where $k \geq 2$. For any integrable functions $f_i : \mathbb{R}^n_+ \to \mathbb{R}_+$, $i = 1, \dots, k$, satisfying S_j-polarity condition with respect to some decreasing function $\rho : \mathbb{R} \to [0, \infty]$, it holds

$$\prod_{i=1}^{k} \int_{\mathbb{R}^n_+} f_i(x_i)\, dx_i \leq \left(\int_{\mathbb{R}^n_+} \rho\left(\binom{k}{j} \|u\|_j^j \right)^{\frac{1}{k}} du \right)^k .$$

- One can replace $\mathbb{R}^n_+$ by $\mathbb{R}^n$ if $f_1, \dots, f_k$ are unconditional.

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Proof of equivalence

- "$\Leftarrow$" trivial.
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- $\Rightarrow$ (if j-Santaló conjecture holds)

$$\begin{aligned} |\lambda K_1(r_1)| \cdots |\lambda K_k(r_k)| &\leq |\mathrm{conv}(\lambda K_1(r_1))| \cdots |\mathrm{conv}(\lambda K_k(r_k))| \\ &\leq |B_j^n|^k. \end{aligned}$$

- $\Rightarrow$

$$(|K_1(r_1)| \dots |K_k(r_k)|)^{1/k} \leq \binom{k}{j}^{-\frac{n}{j}} |B_j^n| \rho^{-1} (r_1 \cdots r_k)^{\frac{kn}{j}}.$$

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$$\begin{aligned} \prod_{i=1}^{k} \int_{\mathbb{R}^n} f_i(x_i)\, dx_i &= \prod_{i=1}^{k} \int_0^\infty |K_i(r_i)|\, dr_i \\ &\leq \binom{k}{j}^{-\frac{kn}{j}} |B_j^n|^k \left(\int_0^\infty \rho^{-1} (r^k)^{\frac{n}{j}}\, dr \right)^k. \end{aligned}$$

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On the other hand,

$$\begin{aligned}
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=& \int_0^\infty \left|\left\{u : \rho\left(\binom{k}{j}\|u\|_j^j\right) \geq t^k\right\}\right| dt \\
=& \int_0^\infty \left|\left\{u : \|u\|_j \leq \left(\binom{k}{j}^{-1}\rho^{-1}(t^k)\right)^{\frac{1}{j}}\right\}\right| dt \\
=& \binom{k}{j}^{-\frac{n}{j}} |B_j^n| \int_0^\infty \rho^{-1}(t^k)^{\frac{n}{j}}\, dt. \quad \square
\end{aligned}$$

- For $x \in \mathbb{R}^n$, write $x = (\widetilde{x}, r)$, where $\widetilde{x} \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}$.
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- For symmetric convex bodies $K_1, K_3, \ldots, K_k$, set $(K_1, K_3, \ldots, K_k)_j^o$:

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  That is, the largest convex set S, s.t. $K_1, S, K_3, \ldots, K_k$ satisfy $\mathcal{E}_j$-polarity condition.

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- Since Steiner symmetrization preserves volume, it suffices to prove that $|K_2| \leq |K_2'|$.
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Let $\tilde{x}_2 \in K_2(r)$ and $\tilde{x}_2' \in K_2(-r)$. Then, for all $(\tilde{x}_i, r_i) \in K_i$, $i = 3, \dots, k$, and for all $(\tilde{x}_1, r_1), (\tilde{x}_1, r_1') \in K_1$, it holds

$$\mathcal{S}_k((\tilde{x}_1, r_1), (\tilde{x}_2, r), (\tilde{x}_3, r_3), \dots, (\tilde{x}_k, r_k)) \leq \binom{k}{k} = 1 \qquad (4)$$

and

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Ball's functional for many sets

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Ball's functional for many sets-some results

- It is equivalent to Ball's conjecture if $k = 2$.
- Holds in the unconditional case.
- Implies the j-Santaló conjecture.

Thank you!!!!!!

Thank you for your attension!!!!!!