Stability and equality cases for the Gaussian (B) inequality

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Part 1: Statement of the main theorems

The (B) inequality

Let γ denote the standard Gaussian in $\mathbb{R}^n$, $\frac{\mathrm{d}\gamma}{\mathrm{d}x} = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$. For us a convex body K is a closed convex set. K is symmetric if $K = -K$.
Given $(t_1, t_2, \dots, t_n) = t \in \mathbb{R}^n$ we define

$$e^t K = \left\{ \left(e^{t_1} x_1, e^{t_2} x_n, \dots, e^{t_n} x_n \right) : \ (x_1, \dots, x_n) \in K \right\}.$$

Theorem (Cordero-Erausquin, Fradelizi, Maurey)

For every symmetric convex body $K \subseteq \mathbb{R}^n$ and $t, s \in \mathbb{R}^n$

$$\gamma \left(e^{\frac{t+s}{2}} K \right) \geq \sqrt{\gamma(e^t K) \gamma(e^s K)}$$

(the strong (B) inequality). In particular, for all $a, b > 0$ we have

$$\gamma \left(\sqrt{ab} K \right) \geq \sqrt{\gamma(aK) \gamma(bK)}$$

(the weak (B) inequality).

Some History

$$\gamma\left(e^{\frac{t+s}{2}}K\right) \geq \sqrt{\gamma(e^tK)\gamma(e^sK)}$$

$$\gamma\left(\sqrt{ab}K\right) \geq \sqrt{\gamma(aK)\gamma(bK)}$$

- The weaker inequality $\gamma\left(\frac{a+b}{2}K\right) \geq \sqrt{\gamma(aK)\gamma(bK)}$ is a corollary of Brunn–Minkowski (Prékopa, Leindler, Borell).
- The weak (B) inequality was originally conjectured by Banaszczyk and popularized by Latała.
- Nayar and Tkocz showed that the symmetry assumption is necessary.
- It is natural to ask if γ can be replaced by other measures. It is conjectured that every even log-concave measure satisfy the weak inequality (μ is log-concave if $\frac{\mathrm{d}\mu}{\mathrm{d}x} = e^{-V}$, V convex). This conjecture is known in dimension $n = 2$ (Böröczky–Lutwak–Yang–Zhang, Saroglou, Livne Bar-On).
- In general dimension, Eskenazis, Nayar and Tkocz proved that certain Gaussian mixtures satisfy the strong (B) inequality.
- With Cordero-Erausquin we proved in particular that rotation invariant log-concave measures satisfy the strong (B) inequality.

Equality case – the weak version

For today we study only the Gaussian case, and the question is when do we have equality in the (B) inequality.

Theorem (Herscovici–Livshyts–R.–Volberg)

Let K be a symmetric convex body, and suppose

$$\gamma\left(\sqrt{ab}K\right) = \sqrt{\gamma(aK)\gamma(bK)}$$

for $b > a > 0$. Then either $K = \mathbb{R}^n$, or K has an empty interior.

One cannot deduce this theorem directly from the proof of Cordero-Erausquin–Fradelizi–Maurey, as their proof includes an approximation step. It can be adopted to yield the equality case when K is smooth (probably), but not for general K. Instead, we prove a stronger stability theorem.

Stability theorem – the weak version

We denote by $r(K)$ the inradius of K , i.e. the largest number $r > 0$ such that $rB_2^n \subseteq K$.

Theorem (Herscovici–Livshyts–R.–Volberg)

Let K be a symmetric convex body, and suppose that

$$\gamma\left(\sqrt{ab}K\right) \leq (1+\varepsilon)\sqrt{\gamma(aK)\gamma(bK)}$$

for $b > a > 0$ and $\varepsilon > 0$ small enough. Then either $r(K) \geq \varphi_{a,b,n}(\varepsilon)$ or $r(K) \leq \frac{1}{\varphi_{a,b,n}(\varepsilon)}$, for an explicit function $\varphi_{a,b,n}$ such that

$$\lim_{\varepsilon \to 0^+} \varphi_{a,b,n}(\varepsilon) = \infty.$$

When proving such a theorem one may assume K is smooth. The equality case then follows by letting $\varepsilon \to 0$.

Quantitative estimates

A more precise statement: If

$$\gamma\left(\sqrt{ab}K\right) \le (1+\varepsilon)\sqrt{\gamma(aK)\gamma(bK)}$$

then either

$$r(K) \ge \frac{1}{b}\sqrt{\log\left(\frac{c\log\left(b/a\right)^2}{n^2\varepsilon}\right)}$$

or

$$r(K) \le \frac{C\sqrt{n}}{a}\log\left(\frac{b}{a}\right)^{-\frac{2}{n+1}}\varepsilon^{\frac{1}{n+1}}.$$

The lower bound $r \gtrsim \sqrt{\log\frac{1}{\varepsilon}}$ is actually sharp, as can be seeing by taking K to be a strip. The upper bound $r \lesssim \varepsilon^{\frac{1}{n+1}}$ is probably not sharp, but one cannot do better then $r \lesssim \sqrt{\varepsilon}$.

Equality case – the strong version

For the strong version one has to be a bit more careful, even with the equality case:

Theorem (Herscovici–Livshyts–R.–Volberg)

Let K be a symmetric convex body, and suppose

$$\gamma\left(e^{\frac{t+s}{2}}K\right) = \sqrt{\gamma(e^{t}K)\gamma(e^{s}K)}$$

for $t, s \in \mathbb{R}^n$. Define

$$H_{t,s} = \operatorname{span}\{e_i : \ 1 \le i \le n \text{ and } t_i = s_i\}.$$

Then either K has an empty interior, or (more generally) $K = K_0 \times H_{t,s}^{\perp}$ for $K_0 \subseteq H_{t,s}$.

So in particular if $t_i \ne s_i$ for all $1 \le i \le n$ then either K has an empty interior or $K = \mathbb{R}^n$.

Stability case – the strong version

To prove a stability theorem for the strong (B) inequality, we need a way to express the idea that "K is close to being a cylinder".

Recall that if K is a convex body with non-empty interior, then at almost every point $x \in \partial K$ there exists a unique supporting hyperplane to K at x, and we denote the normal to this hyperplane by n_x.

Lemma

Fix a subspace $H \subseteq \mathbb{R}^n$. Assume K has non-empty interior, and that $n_x \in H$ for almost every $x \in \partial K$. Then there exists a convex body $K_0 \subseteq H$ such that $K = K_0 \times H^{\perp}$.

Stability case – the strong version

Theorem (Herscovici–Livshyts–R.–Volberg, Informal Version)

Let K be a symmetric convex body, and suppose

$$\gamma\left(e^{\frac{t+s}{2}}K\right) \leq (1+\varepsilon)\sqrt{\gamma(e^{t}K)\gamma(e^{s}K)}$$

for $t, s \in \mathbb{R}^n$. Then either:

1. *$r(K)$ is “large”.*
2. *$r(K)$ is “small”.*
3. *Define*

$$H_{t,s,\delta} = \operatorname{span}\left\{e_i : \ 1 \leq i \leq n \text{ and } |t_i - s_i| < \delta\right\}.$$

Then at “most” points $x \in \partial K$ (in the sense of measure), the normal n_x is “almost” in $H_{t,s,\delta}$ (in the sense that $\operatorname{Proj}_{H^{\perp}_{t,s,\delta}}(n_x) \approx 0$).

Part 2: An application – Maximal Gaussian Measure position

Maximal Gaussian measure position

Let K be a symmetric convex body, compact with non-empty interior. We say that K is in Maximal Gaussian Measure position if

$$\gamma(K) = \min\left\{\gamma(T(K)) : \; T \in SL(n)\right\}.$$

This is an interesting position for two reasons:

Theorem (Bobkov)

K is in Maximal Gaussian Measure position if and only if the measure

$$\gamma_K(A) = \frac{\gamma\left(A \cap K\right)}{\gamma(K)}$$

is isotropic, i.e. $\int x_i x_j \mathrm{d}\gamma_K = C \cdot \delta_{ij}$ for a constant $C > 0$.

Maximal Gaussian Measure position – Contd.

$$\boxed{\gamma(K) = \min\left\{\gamma(T(K)) :\ \ T \in SL(n)\right\}}$$

Theorem (Bobkov)

If K is in Maximal Gaussian Measure position, then it is in M-position. Explicitly, let D be the ball with $|K| = |D|$, then

$$\begin{array}{ll} |K \cap D| \geq C^{-n}\,|D| & |K^\circ \cap D| \geq C^{-n}\,|D| \\ |K + D| \leq C^{n}\,|D| & |K^\circ + D| \leq C^{n}\,|D| \end{array}$$

for an absolute constant $C > 0$.

The M-position is very useful in Asymptotic Geometric Analysis, but is very non-unique. The Maximal Gaussian Measure position can be a canonical choice for an M-position, if it is unique.

Uniqueness of Maximal Gaussian Measure position

Corollary (Of our theorem, see also Artstein–Katzin and Artstein–Putterman)

The Maximal Gaussian Measure position of a symmetric convex body is unique up to rotations.

Proof.

If not, there exists a convex body K and a vector $x \neq 0$ such that $|e^x K| = |K|$ and both K and $e^x K$ are in Maximal Gaussian Measure position. By the (B) inequality

$$\gamma\left(e^{\frac{x}{2}} K\right) \geq \sqrt{\gamma(K)\gamma\left(e^x K\right)}, \qquad (☺)$$

so $e^{\frac{x}{2}} K$ is also in Maximal Gaussian Measure position. But this means that we have equality in (☺), which is impossible by our theorem.

□

Part 3: Stability in Poincaré inequalities

On the proof of Cordero–Fradelizi–Maurey

The weak (B) inequality,

$$\gamma\left(\sqrt{ab}K\right) \geq \sqrt{\gamma(aK)\gamma(bK)},$$

just means that $\rho(t) = \log \gamma\left(e^t K\right)$ is concave. The condition $\rho''(0) \leq 0$ turns out to be the same as

$$\int |x|^4 \, \mathrm{d}\gamma_K - \left(\int |x|^2 \, \mathrm{d}\gamma_K\right)^2 \leq 2 \int |x|^2 \, \mathrm{d}\gamma_K$$

(recall that $\gamma_K(A) = \frac{\gamma(A \cap K)}{\gamma(K)}$). Something more general is true:

Theorem (Cordero–Fradelizi–Maurey)

For every symmetric convex body K and every even function $f : \mathbb{R}^n \to \mathbb{R}$,

$$\int f^2 \mathrm{d}\gamma_K - \left(\int f \, \mathrm{d}\gamma_K\right)^2 \leq \frac{1}{2} \int |\nabla f|^2 \, \mathrm{d}\gamma_K.$$

Stability in the even Poincaré inequality

Our stability theorem follows from

Theorem

Let K be a symmetric convex body, and assume that

$$\int |x|^4 \,\mathrm{d}\gamma_K - \left(\int |x|^2 \,\mathrm{d}\gamma_K\right)^2 \geq 2\int |x|^2 \,\mathrm{d}\gamma_K - \varepsilon.$$

Then either $r(K) \geq \sqrt{\log \frac{c}{n^2\varepsilon}}$ *or* $r(K) \leq C\sqrt{n}\varepsilon^{\frac{1}{n+1}}$.

For the stability of the strong (B) theorem we have a similar theorem for $\langle Tx, x\rangle$ instead of $|x|^2$, and then the bounds depend on the smallest singular value of T. We do not have (and do not need) stability of the even Poincaré inequality for non-quadratic functions.

The proof structure

The even Poincaré inequality

$$\int f^2 \mathrm{d}\gamma_K - \left(\int f \mathrm{d}\gamma_K\right)^2 \leq \frac{1}{2}\int |\nabla f|^2 \, \mathrm{d}\gamma_K$$

is proved by a nowadays standard L^2-argument, which reduces it to the usual Poincaré inequality for non-necessarily-even functions

$$\int g^2 \mathrm{d}\gamma_K - \left(\int g \mathrm{d}\gamma_K\right)^2 \leq \int |\nabla g|^2 \, \mathrm{d}\gamma_K.$$

In the same way, our proof reduces the stability of the even Poincaré inequality to a previous stability result of Livshyts (slightly improved in our paper).

Stability in the usual Poincaré inequality

Theorem (Livshyts (essentially))

Assume for some function $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int g^2 \mathrm{d}\gamma_K - \left(\int g \mathrm{d}\gamma_K \right)^2 \geq \int |\nabla g|^2 \mathrm{d}\gamma_K - \varepsilon.$$

Then there exists a linear function $\ell(x) = \langle x, \theta \rangle + v$ such that:

1. $\|g - \ell\|_{W^{1,2}(\gamma_K)} \leq 4\varepsilon$.
2. $\int_{\partial K} \langle n_x, \theta \rangle^2 \mathrm{d}\gamma_{\partial K} \leq \frac{2(n+1)\gamma(K)}{r(K)} \cdot \varepsilon$.

Here $\gamma_{\partial K}$ is the measure on ∂K with density $(2\pi)^{-\frac{n}{2}} e^{-|x|^2/2}$ with respect to the $(n-1)$-dimensional Hausdorff measure.

The isoperimetric inequality

We will not explain the L^2 proof in more details. Instead, we mention that we do not conclude directly that $r(K)$ is "very large" or "very small". Instead, we obtain

$$\frac{\gamma(K)}{\int_{r(K)B_2^n} |x|^2 \, \mathrm{d}\gamma} + \frac{\gamma(K)}{r(K)\gamma^+(\partial K)} \geq \frac{c}{n^2 \varepsilon},$$

and this condition should be analyzed. To bound the second term, the Gaussian isoperimetric inequality makes a surprising appearance.

Proposition

If $\gamma(K) \geq \frac{1}{2}$ then

$$\gamma^+(\partial K) \geq \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} r(K)^2}.$$

If $\gamma(K) \leq \frac{1}{2}$ then

$$\gamma^+(\partial K) \geq \left(\frac{c \cdot r(K)}{\sqrt{n}} \right)^n e^{-\frac{1}{2} r(K)^2}.$$

Part 4: Coffee Break!