Random Polytopes in Polytopes

Cortona, June 2023

Matthias Reitzner

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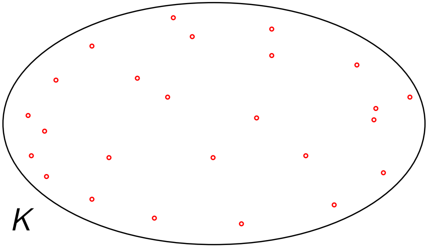
Random polytopes

Random points $X_1, \ldots, X_n$ uniformly in $K \subset \mathbb{R}^d$

$$P_n = [X_1, \ldots, X_n] \subset K$$



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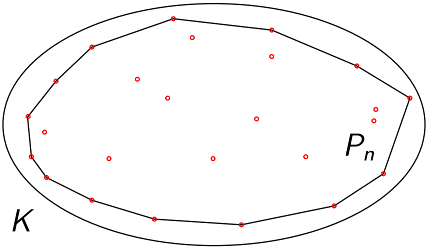

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Random points $X_1, \dots, X_n$ uniformly in $K \subset \mathbb{R}^d$

$$P_n = [X_1, \dots, X_n] \subset K$$

- $\vec{f}$-vector: $\vec{f}(P_n) = \begin{pmatrix} f_0(P_n) \\ f_1(P_n) \\ \vdots \\ f_{d-1}(P_n) \end{pmatrix} = \begin{pmatrix} \sharp \ \text{vertices of } P_n \\ \sharp \ \text{edges of } P_n \\ \vdots \\ \sharp \ \text{facets of } P_n \end{pmatrix}$

- volume of P_n : $V_d(P_n)$

A) Expectations

A) Expectations: $K \in \mathcal{K}^d_{2+}$

$K \in \mathcal{K}^d_{2,+}$:

$$\mathbb{E}\, \vec{\boldsymbol{f}}(P_n) = \vec{\boldsymbol{c}}\; \Omega_d(K)\, n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}})$$

$$V_d(K) - \mathbb{E}\, V_d(P_n) = c_d\, \Omega_d(K)\, n^{-\frac{2}{d+1}} + o(n^{-\frac{2}{d+1}})$$

Bárány, Buchta, Efron, R. , Rényi, Schneider, Sulanke, Wieacker, ...

$$\Omega_d(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{d+1}}\, dx$$

A) Expectations: $K \in \mathcal{K}^d_{2+}$

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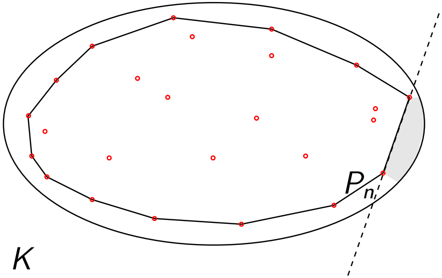
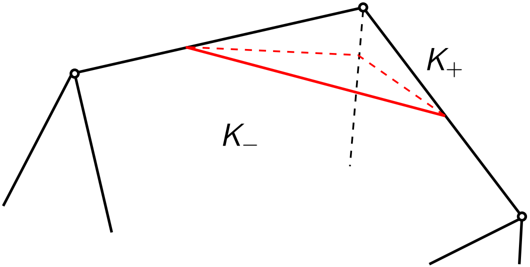


A) Expectations: $K \in \mathcal{P}^d$

$$\mathbb{E}(V_d(K) - V_d(P_n)) \approx \binom{n}{d} \int V_d(K_-)^{n-d} V_{diff}(K_+) V_{d-1}(K \cap H)^{d+1} dH$$



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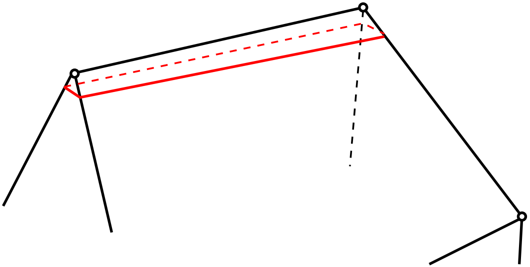


A) Expectations: $K \in \mathcal{P}^d$

$K \in \mathcal{P}^d$:

$$\mathbb{E}\,\vec{\boldsymbol{f}}(P_n) = \tilde{\vec{\boldsymbol{c}}}\ T(K)\,\ln^{d-1} n + o(\ln^{d-1} n)$$

$$V_d(K) - \mathbb{E}\,V_d(P_n) = \tilde{c}_d\,T(K)\,n^{-1}\ln^{d-1} n + o(n^{-1}\ln^{d-1} n)$$

Affentranger, Bárány, Buchta, Efron, R. , Schneider , Schütt, ...

A) Expectations in $\mathbb{R}^2$

$K \in \mathcal{P}^2$ with ℓ vertices:

$$\mathbb{E}f_0(P_n) = \mathbb{E}f_1(P_n) = \frac{2\ell}{3}\log n + \frac{2}{3}\sum_{i=1}^{\ell}\log\left(\frac{F_i}{V_2(P)}\right) + \frac{2\gamma\ell}{3} + O(n^{-\frac{1}{4}}(\log n)^2)$$

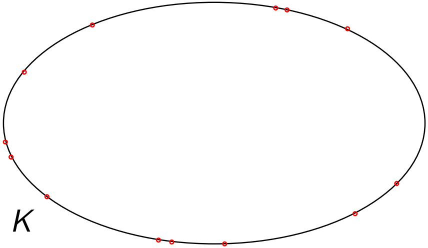
$$V_2(K) - \mathbb{E}V_2(P_n) = \frac{2\ell}{3}(\log n)n^{-1} + \left[\frac{2}{3}\sum_{i=1}^{\ell}\log\left(\frac{F_i}{V_2(P)}\right) + \frac{2\gamma\ell}{3}\right]n^{-1} + O(n^{-\frac{5}{4}}(\log n)^2).$$

Renyi & Sulanke, Gusakova & R & Thäle

A) Random polytopes with vertices on ∂K

random points $X_1, \ldots, X_n$ uniformly on ∂K

$$P_n = [X_1, \ldots, X_n] \subset K$$



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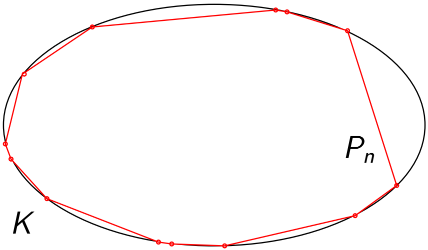

A) Expectations

$K \in \mathcal{K}^d_{2,+}$:

$$\mathbb{E}\, f_0(P_n) = n$$

$$V_d(K) - \mathbb{E} V_d(P_n) = c_d \int_{\partial K} \kappa(x)^{\frac{1}{d-1}}\, dx\, n^{-\frac{2}{d-1}} + o(n^{-\frac{2}{d-1}})$$

Affentranger, Böröczky, Buchta, Fodor, Hug, Gruber, Müller, R. , Schneider, Schütt, Tichy, Werner, . . .

A) Expectations

$$K \in \mathcal{K}^d_{1+\epsilon,+}: \quad \mathbb{E}\, f_0(P_n) = n$$

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$d \geq 2 : \quad f_0(P_n) = n$

$d = 2 : \quad f_0(P_n) = f_1(P_n) = n$

$d = 3 : \quad f_0(P_n) = n, \; f_1(P_n) = 3n - 6, \; f_2 = 2n - 4$

A) Expectations

$K \in \mathcal{K}^d_{1+\epsilon,+}$: $\mathbb{E}\, f_0(P_n) = n$

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$d \geq 2$: $f_0(P_n) = n$

$d = 2$: $f_0(P_n) = f_1(P_n) = n$

$d = 3$: $f_0(P_n) = n,\ f_1(P_n) = 3n - 6,\ f_2 = 2n - 4$

$d \geq 4$: $f_i(P_n)$ is a random variable, $i \geq 1$ — Stemeseder

A) Expectations

$K \in \mathcal{K}^d_{2,+}$:
$$\mathbb{E}\, f_0(P_n) = n$$
$$V_d(K) - \mathbb{E}\, V_d(P_n) = c_d \int_{\partial K} \kappa(x)^{\frac{1}{d-1}}\, dx\, n^{-\frac{2}{d-1}} + o(n^{-\frac{2}{d-1}})$$

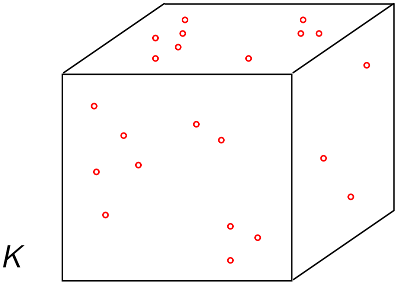
$$\mathbb{E}\, \vec{f}(P_n) = \vec{c}(K)\, n + \cdots$$

Stemeseder

A) Random polytopes with vertices on ∂K

$K \in \mathcal{P}^d$: random points $X_1, \ldots, X_n$ uniformly on ∂K

$$P_n = [X_1, \ldots, X_n] \subset K$$



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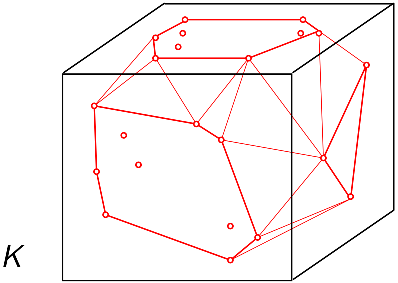

P_n neither simple nor simplicial

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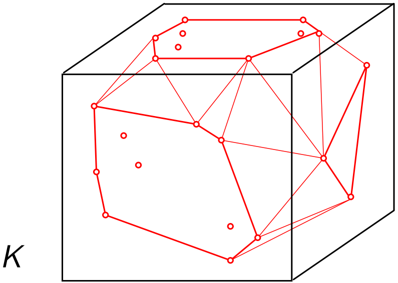

$\mathbb{E} f_0(P_n) = ?$

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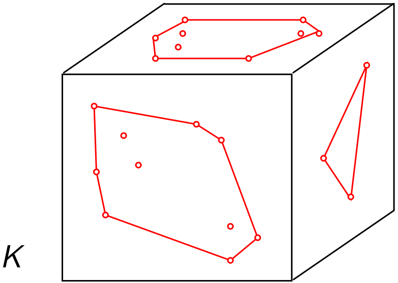

$\mathbb{E} f_0(P_n) = ?$

P_n neither simple nor simplicial

A) Expectations

$K \in \mathcal{P}^d$:

$$\mathbb{E} f_0(P_n) = c_d \, T(K) \ln^{d-1} n + \cdots$$

Wieacker, Bárány & Buchta

A) Expectations

$K \in \mathcal{P}^d$:

$$\mathbb{E} f_0(P_n) = c_d\, T(K)\, \ln^{d-1} n + \cdots$$

Wieacker, Bárány & Buchta

$$F \in \mathcal{F}_{d-1}(K): \quad f_0(P_n \cap F) = c_{d-1}\, T(F)\, \ln^{d-2}\left(n \frac{V_{d-1}(F)}{S(K)}\right) + \cdots$$

A) Expectations

$K \in \mathcal{P}^d$:

$$\mathbb{E} f_0(P_n) = c_d \, T(K) \ln^{d-1} n + \cdots$$

Wieacker, Bárány & Buchta

$$F \in \mathcal{F}_{d-1}(K): \quad f_0(P_n \cap F) = c_{d-1} \, T(F) \ln^{d-2} \; n \qquad\qquad + \cdots$$

A) Expectations

$K \in \mathcal{P}^d$:

$$\mathbb{E}\, f_0(P_n) = c_{d-1} \underbrace{\sum_{\mathcal{F}_{d-1}(K)} T(F)}_{=T(K)} \ln^{d-2} n + \cdots$$

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$K \in \mathcal{P}^d$:

$$\mathbb{E}\, f_0(P_n) = c_{d-1} \underbrace{\sum_{\mathcal{F}_{d-1}(K)} T(F)}_{=T(K)} \ln^{d-2} n + \cdots$$

- $\mathbb{E} f_{d-1}(P_n) = ?$
- $V_d(K) - \mathbb{E} V_d(P_n) = ?$

A) Expectations

$K \in \mathcal{P}^d$ a simple polytope:

$$
\begin{aligned}
\mathbb{E}\, f_0(P_n) &= c_d\, T(K) \ln^{d-2} n \quad + o(\ln^{d-2} n) \\
\mathbb{E} f_{d-1}(P_n) &= c_d\, T(K) \ln^{d-2} n \quad + o(\ln^{d-2} n) \\
V_d(K) - \mathbb{E} V_d(P_n) &= c(K) n^{-\frac{d}{d-1}} \quad + o(n^{-\frac{d}{d-1}})
\end{aligned}
$$

R & Schütt & Werner

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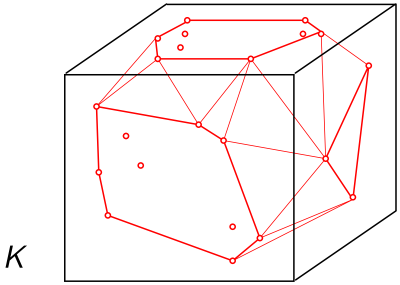


A) Expectations

$K \in \mathcal{P}^d$ a simple polytope:

$$\begin{aligned}
\mathbb{E}\, f_0(P_n) &= c_d\, T(K)\, \ln^{d-2} n \quad + o(\ln^{d-2} n) \\
\mathbb{E} f_{d-1}(P_n) &= c_d\, T(K) \ln^{d-2} n \quad + o(\ln^{d-2} n) \\
V_d(K) - \mathbb{E}\, V_d(P_n) &= c(K) n^{-\frac{d}{d-1}} \quad + o(n^{-\frac{d}{d-1}})
\end{aligned}$$

R & Schütt & Werner

$\Rightarrow$ general $K \in \mathcal{P}^d$

$\Rightarrow$ general $K_m \in \mathcal{P}^d$, $n, m \to \infty$

B) Central Limit Theorems

B) CLT

Theorem

For $K \in \mathcal{K}^d_{2+}$ or K a polytope:

$$\left| \mathbb{P}\left(\frac{f_i(P_n) - \mathbb{E} f_i(P_n)}{\sqrt{\mathbb{V} f_i(P_n)}} \le x \right) - \Phi(x) \right| \to 0$$

for all $i = 0, \dots, d-1$, and

$$\left| \mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E} V_d(P_n)}{\sqrt{\mathbb{V} V_d(P_n)}} \le x \right) - \Phi(x) \right| \to 0$$

as $t \to \infty$.

Cabo, Groeneboom, Hsing, Bräker, Bárány, Reitzner, Pardon, Calka, Schreiber, Schulte, Yukich, Thäle, Lachieze-Rey,

B) CLT for $\mathcal{K}^d_{2+}$

Reitzner, Vu

K sufficiently smooth:

$$\left| \mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E}V_d(P_n)}{\sqrt{\mathbb{V}V_d(P_n)}} \leq x \right) - \Phi(x) \right| \leq c(K) \underbrace{n^{-\frac{1}{2}+\frac{1}{d+1}}}_{(\mathbb{V}V_d(P_n))^{-\frac{1}{2}}} \ln^{2+\frac{2}{d+1}} n$$

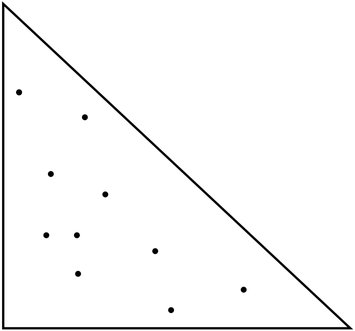
B) CLT for $\mathcal{K}^d_{2+}$

Reitzner, Vu, Lachieze-Rey & Schulte & Yukich

K sufficiently smooth:

$$\left| \mathbb{P}\left(\frac{V_d(P_n) - \mathbb{E} V_d(P_n)}{\sqrt{\mathbb{V} V_d(P_n)}} \leq x \right) - \Phi(x) \right| \leq c(K) \underbrace{n^{-\frac{1}{2}+\frac{1}{d+1}}}_{(\mathbb{V} V_d(P_n))^{-\frac{1}{2}}}$$

B) CLT for $\mathcal{P}^2$



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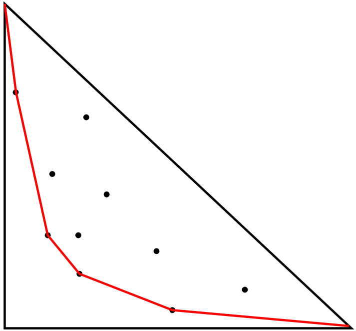
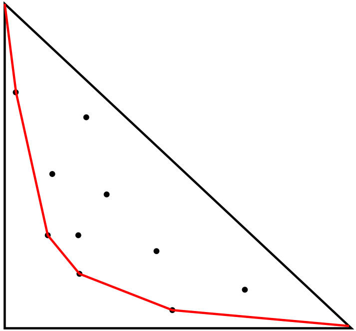


$$\mathbb{P}(f_0(T_m) = k) = 2^k \sum_{\sum i_j = m} \frac{1}{i_1(i_1+i_2)\dots(i_1+\dots+i_k)} \frac{i_1 \dots i_k}{(i_1+1)(i_1+i_2+1)\dots(i_1+\dots+i_k+1)}$$ Buchta

B) CLT for $\mathcal{P}^2$



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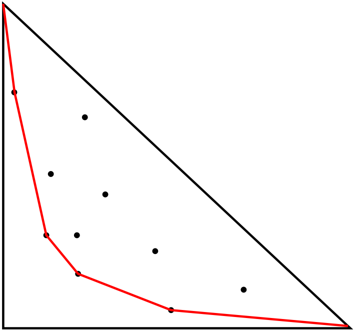
Buchta

$$G_n(z) = \frac{2^n}{n!(n+1)!} \prod_1^m (z - z_k)$$

B) CLT for $\mathcal{P}^2$



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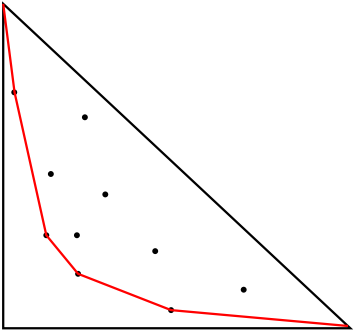
$$G_n(z) = \frac{2^n}{n!(n+1)!} \prod_1^m (z - z_k)$$
$$f_0(T_m) = 1 + B_1 + \dots + B_m$$

B) CLT for $\mathcal{P}^2$

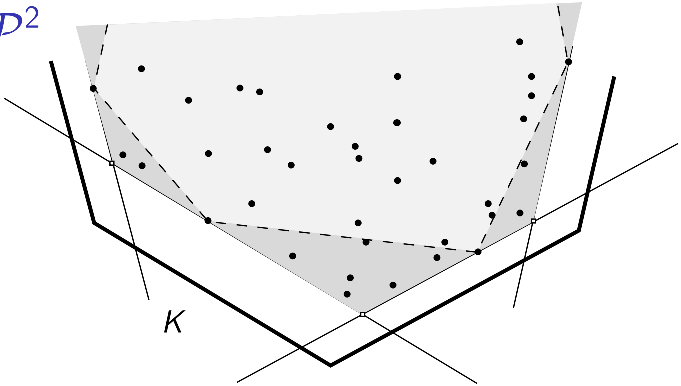


$$\mathbb{P}(f_0(T_m) = k) = 2^k \sum_{\sum i_j = m} \frac{1}{i_1(i_1+i_2)\dots(i_1+\dots+i_k)} \frac{i_1 \dots i_k}{(i_1+1)(i_1+i_2+1)\dots(i_1+\dots+i_k+1)}$$ Buchta

$$G_n(z) = \frac{2^n}{n!(n+1)!} \prod_1^m (z - z_k)$$
$$f_0(T_m) = 1 + B_1 + \dots + B_m$$

$$\left| \mathbb{P}\left(\frac{f_0(T_m) - \mathbb{E} f_0(T_m)}{\sqrt{\mathbb{V} f_0(T_m)}} \le x \right) - \Phi(x) \right| \le \frac{c}{\sqrt{\log m}},$$ Gusakova & Thäle

B) CLT for $\mathcal{P}^2$



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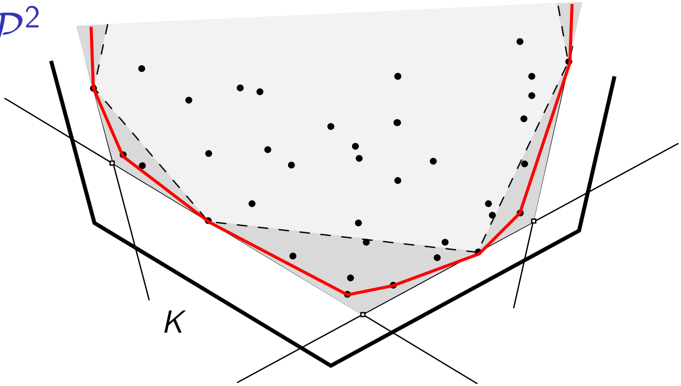

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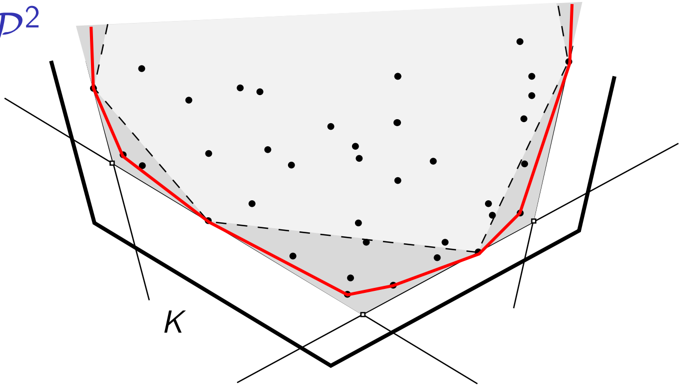


$K \in \mathcal{P}^2$:

$$\left| \mathbb{P}\left(\frac{f_0(P_n) - \mathbb{E} f_0(P_n)}{\sqrt{\mathbb{V} f_0(P_n)}} \le x \right) - \Phi(x) \right| \le \frac{c}{\sqrt{\log n}},$$

Gusakova & R & Thäle

B) CLT for $\mathcal{K}^d$: rd points on ∂K

Theorem

For $K \in \mathcal{K}^d_{2+}$:

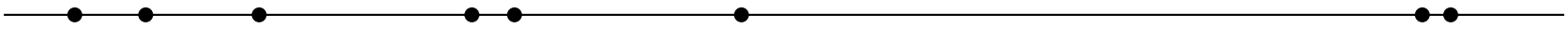
$$\left| \mathbb{P}\left(\frac{V_d(\mathbb{P}_n) - \mathbb{E} V_d(\mathbb{P}_n)}{\sqrt{\mathbb{V} V_d(\mathbb{P}_n)}} \le x \right) - \Phi(x) \right| \to 0$$

Vu & Wu, Thäle, Stemeseder

- CLTs for $f_i(\mathbb{P}_n)$ for $i = 1, \dots, d-1$
- CLTs if K is a polytope?

C) Order Statistic

C) Order statistic



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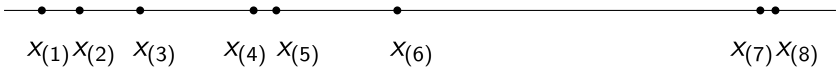

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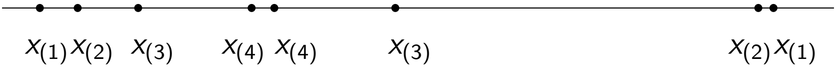
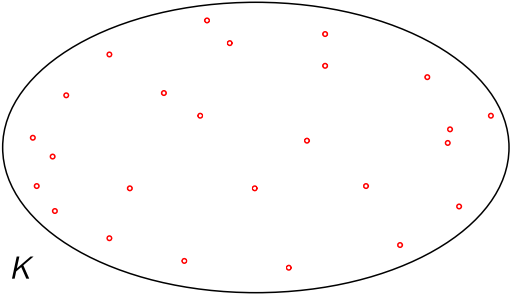


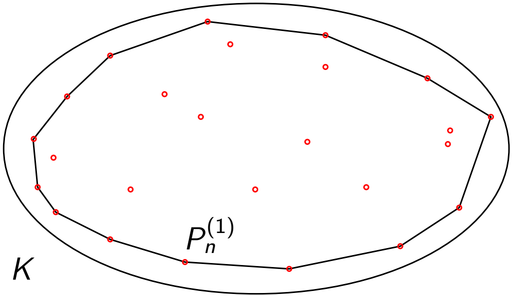
C) Convex hull peeling

random points



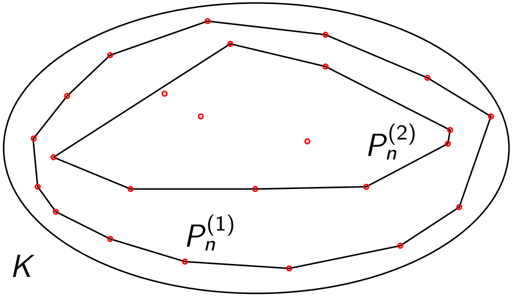
C) Convex hull peeling

convex hull of random points



C) Convex hull peeling

“order statistic”



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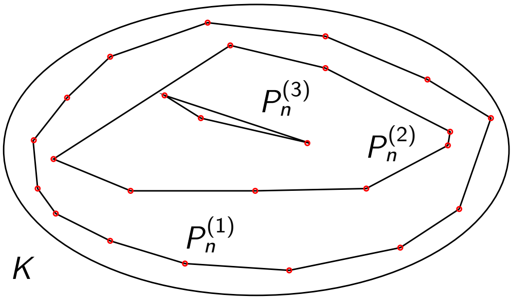
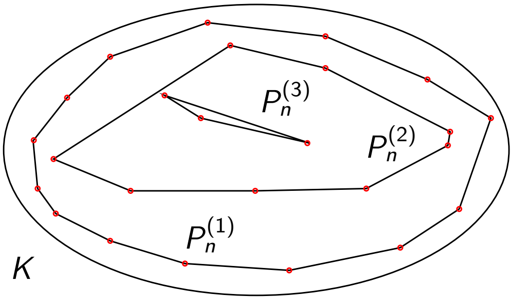


C) Convex hull peeling

“order statistic”



$$L_n(x) = \max\{k \colon x \in P_n^{(k)}\}$$

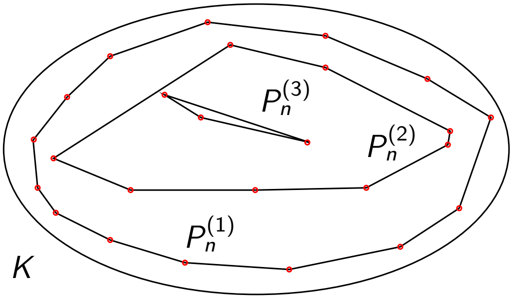
number of layers $\quad L_n = \max_x L_n(x) = \max\{k \colon P_n^{(k)} \neq \emptyset\}$

C) Convex hull peeling

Random points $X_1, \ldots, X_n$ uniformly in $K \subset \mathbb{R}^d$

- $\vec{f}(P_n^{(\ell)}) = \begin{pmatrix} f_0(P_n^{(\ell)}) \\ f_1(P_n^{(\ell)}) \\ \vdots \\ f_{d-1}(P_n^{(\ell)}) \end{pmatrix}$
- $V_d(P_n^{(\ell)}) =$ volume of $P_n^{(\ell)}$
- number of layers $\quad L_n = \max\{k \colon P_n^{(k)} \neq \emptyset\}$

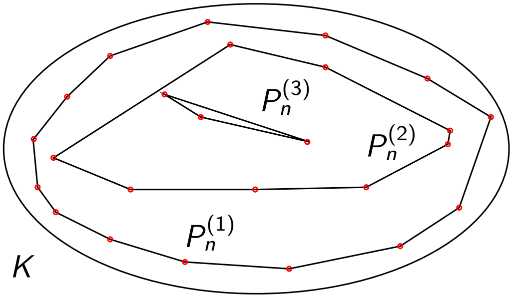
C) Convex hull peeling



$$\mathbb{E} f_0(P_n^{(1)}) = c_d \Omega_d(K)\, n^{\frac{d-1}{d+1}} + \cdots$$

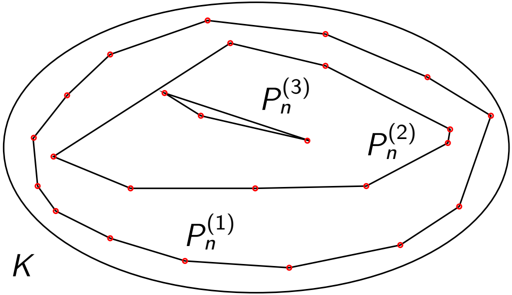
$$\mathbb{E} f_0(P_n^{(2)}) = c_d \text{ ``}T(P_n^{(1)})\text{``} \ln^{d-1} n + \cdots$$

C) Convex hull peeling



$$\mathbb{E} f_0(P_n^{(1)}) = c_d \Omega_d(K)\, n^{\frac{d-1}{d+1}} + \cdots$$
$$\mathbb{E} f_0(P_n^{(2)}) = c_d \Omega_d(K)\, n^{\frac{d-1}{d+1}} \ln^{d-1} n + \cdots$$

C) Convex hull peeling in B^d

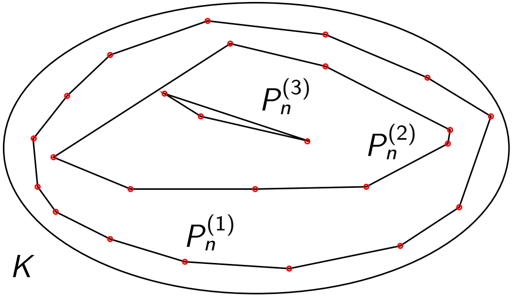


$K = B^d$:

$$\mathbb{E}\vec{f}(P_n^{(\ell)}) = c_{d,j,\ell}\, n^{\frac{d-1}{d+1}} + \cdots$$

Calka & Quilan: Poisson PP, CLT, ...

C) Convex hull peeling

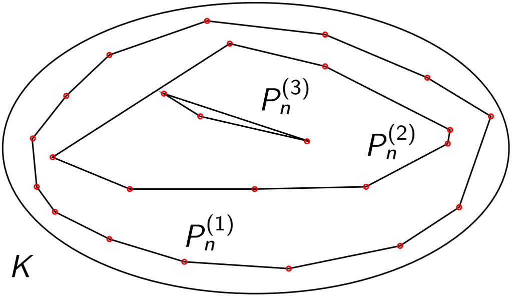


$K \in \mathcal{K}^d$:

$$c_1 n^{\frac{2}{d+1}} \leq \mathbb{E} L_n \leq c_2 n^{\frac{2}{d+1}}$$

Dalal

C) Convex hull peeling



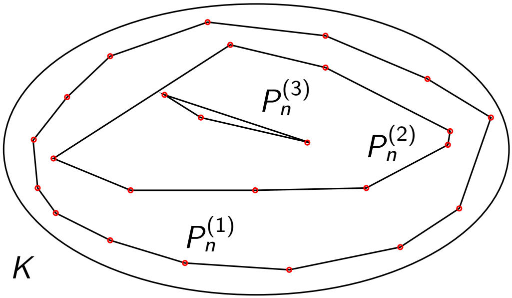
$K \in \mathcal{K}^d$:

$$c_1 n^{\frac{2}{d+1}} \leq \mathbb{E} L_n \leq c_2 n^{\frac{2}{d+1}}$$

Dalal

$\Rightarrow$: mean layer $\qquad f_0(P_n^{(\ell)}) \approx \frac{n}{n^{\frac{2}{d+1}}} = n^{\frac{d-1}{d+1}}$

C) Convex hull peeling



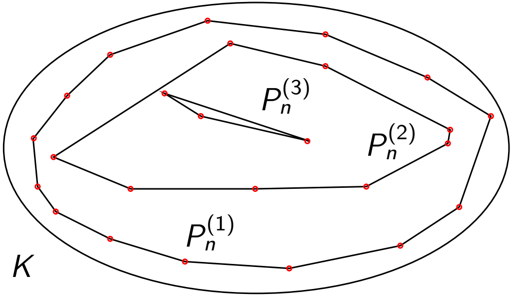
$K \in \mathcal{K}^d$:

$$c_1 \le \mathbb{E} \max \, n^{-\frac{2}{d+1}} L_n(x) \le c_2$$

Dalal

$\Rightarrow$: mean layer $\qquad f_0(P_n^{(\ell)}) \approx \dfrac{n}{n^{\frac{2}{d+1}}} = n^{\frac{d-1}{d+1}}$

C) Convex hull peeling

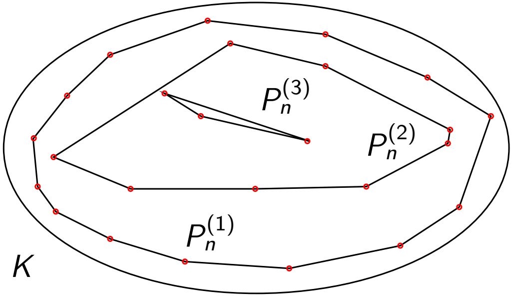


$K \in \mathcal{K}^d$:

$$n^{-\frac{2}{d+1}} L_n(x) \longrightarrow h(x) \qquad \text{in probab}$$

Calder & Smart

C) Convex hull peeling



$K \in \mathcal{K}^d$:

$$n^{-\frac{2}{d+1}} L_n(x) \longrightarrow h(x) \qquad \text{in probab}$$

Calder & Smart

$h(x)$ unique viscosity solution of

$$\langle \nabla h, \mathrm{cof}(-\mathrm{Hess}\, h) \nabla h \rangle = 1 \quad \text{with} \quad h = 0 \text{ on } \partial K$$

Thank you!