An Elementary Question About Functional Intrinsic Volumes

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joint work with Daniel Hug and Jacopo Ulivelli

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The Basic Observation

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Closer Inspection

If K is as above, then $\dim K < n - 1$.

More Generally (Actually Equivalent)

Recall: Steiner Formula

$$\operatorname{vol}_n(K + rB^n) = \sum_{j=0}^{n} r^{n-j} \kappa_{n-j} V_j(K), \quad r > 0, K \in \mathcal{K}^n$$

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Let $1 \leq i \leq j \leq n$. If $K \in \mathcal{K}^n$ is such that $V_i(K) = 0$, then $V_j(K) = 0$.

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If K is as above, then $\dim K < i$.

Local Version

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Recall: Area Measures

$$S_{n-1}(K+rB^n,\omega)=\sum_{j=0}^{n-1} r^{n-1-j}\binom{n-1}{j} S_j(K,\omega) \quad r>0, K\in\mathcal{K}^n, \omega\in\mathcal{B}(\mathbb{S}^{n-1})$$

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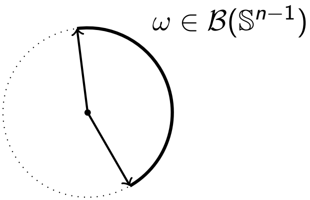

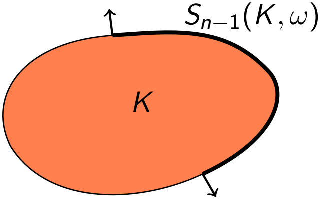

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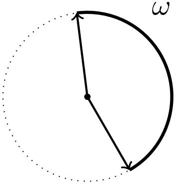
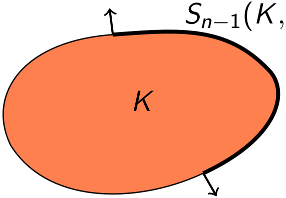
$$V_j(K) = \frac{1}{n\kappa_{n-j}} \binom{n}{j} S_j(K, \mathbb{S}^{n-1}) \qquad 0 \leq j \leq n-1$$





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$$\operatorname{supp} S_j(K,\cdot)\subseteq \operatorname{supp} S_i(K,\cdot), \quad 1\le i\le j<n, K\in\mathcal{K}^n$$

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Theorem (Schneider, Math. Ann. 1975)

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For $K \in \mathcal{K}^n$, let $h_K(x) = \sup_{y\in K}\langle x, y\rangle$. We have $\partial h_K(o) = K$.

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If $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \cap C^2(\mathbb{R}^n)$, then $\mathrm{dMA}(v;x) = \det(\mathrm{D}^2 v(x))\,\mathrm{d}x$.

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- j-hom.: $\overline{\mathrm{V}}_{j,\alpha}^*(\lambda v) = \lambda^j \, \overline{\mathrm{V}}_{j,\alpha}^*(v)$

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Theorem (Hadwiger, 1950s)

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Theorem (Colesanti, Ludwig, M., '20+, Adv. Math. '23; Knoerr '22+)

A map $\mathrm{Z}\colon \mathrm{Conv}(\mathbb{R}^n;\mathbb{R}) \to \mathbb{R}$ *is a continuous, dually epi-translation and rotation invariant valuation if and only if there exist* $\alpha_0, \dots, \alpha_n \in C_c([0,\infty))$ *such that*

$$\mathrm{Z}(v) = \overline{\mathrm{V}}^*_{0,\alpha_0}(v) + \cdots + \overline{\mathrm{V}}^*_{n,\alpha_n}(v)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$.

Elementary Question

Functional Fact of Life?

Let $1 \leq i \leq j \leq n$ and let $\alpha, \beta \in C_c([0,\infty))$ be "suitable". If $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ is such that

$$\overline{\mathrm{V}}_{i,\alpha}^*(v) = 0,$$

does it then follow that also

$$\overline{\mathrm{V}}_{j,\beta}^*(v) = 0?$$

Suitable Densities

Recall: Desired Fact of Life

For $1 \leq i \leq j \leq n$, $\alpha, \beta \in C_c([0,\infty))$: $\overline{\mathrm{V}}^*_{i,\alpha}(v) = 0 \implies \overline{\mathrm{V}}^*_{j,\beta}(v) = 0$

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For $t \geq 0$ let $v_t \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$ be given by

$$v_t(x) = \begin{cases} 0 & \text{if } |x| \leq t, \\ |x| - t & \text{if } |x| > t \end{cases}$$

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Calculations show:

$$\overline{\mathrm{V}}^*_{j,\alpha}(v_t) = \frac{\kappa_n}{\kappa_{n-j}} \binom{n}{j} \alpha(t)$$

Suitable Densities

Recall: Desired Fact of Life

For $1 \leq i \leq j \leq n$, $\alpha, \beta \in C_c([0,\infty))$: $\overline{\mathrm{V}}^*_{i,\alpha}(v) = 0 \implies \overline{\mathrm{V}}^*_{j,\beta}(v) = 0$

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Connection with the Dimension

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Dimension of $v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$

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For t large enough: $\overline{\mathrm{V}}_{j,\alpha}^*(v_t) = 0$ but $\dim v_t = n$.

A Solution

Functional Fact of Life

Let $1 \leq i \leq j \leq n$ and let $\alpha, \beta \in C_c([0,\infty))$ be non-negative with $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$.

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Corollary (Hug, M., Ulivelli, '23+)

Let $1 \leq i \leq j \leq n$ and let $\alpha, \beta \in C_c([0,\infty))$ be non-negative with $\operatorname{supp}\beta \subseteq \operatorname{supp}\alpha$. If $v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R})$ is such that

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A Special Family of Mixed Monge–Ampère Measures

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Mixed Monge–Ampère Measures

For $m \in \mathbb{N}$, $v_1, \ldots, v_m \in \text{Conv}(\mathbb{R}^n; \mathbb{R})$, $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\mathrm{MA}(\lambda_1 v_1 + \cdots + \lambda_m v_m; \cdot) = \sum_{i_1, \ldots, i_n = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \mathrm{MA}(v_{i_1}, \ldots, v_{i_n}; \cdot).$$

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If $v_1, \dots, v_n \in \text{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C^2(\mathbb{R}^n)$, then

$$\mathrm{dMA}(v_1, \dots, v_n; x) = \det(\mathrm{D}^2 v_1(x), \dots, \mathrm{D}^2 v_n(x))\, \mathrm{d}x.$$

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Special Family

$$\mathrm{MA}_j(v; \cdot) = \mathrm{MA}(v[j], h_{B^n}[n-j]; \cdot), \quad v \in \text{Conv}(\mathbb{R}^n; \mathbb{R}), 0 \leq j \leq n$$

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$$= \sum_{j=0}^{n} r^{n-j} \binom{n}{j} \int_{\mathbb{R}^n} \alpha(|x|)\, \mathrm{dMA}_j(v; x)$$

Representation Formula

$$\overline{\mathrm{V}}_{j,\alpha}^{*}(v) = \binom{n}{j} \frac{1}{\kappa_{n-j}} \int_{\mathbb{R}^n} \alpha(|x|)\, \mathrm{dMA}_j(v; x), \quad v \in \operatorname{Conv}(\mathbb{R}^n;\mathbb{R}), 0 \le j \le n$$

A Solution

Local Functional Fact of Life

$$\operatorname{supp} \mathrm{MA}_j(v,\cdot) \subseteq \operatorname{supp} \mathrm{MA}_i(v,\cdot), \quad 1 \leq i \leq j \leq n, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$$

A Solution

Theorem (Hug, M., Ulivelli, '23+)

$$\operatorname{supp} \mathrm{MA}_j(v,\cdot) \subseteq \operatorname{supp} \mathrm{MA}_i(v,\cdot), \quad 1 \le i \le j \le n, v \in \mathrm{Conv}(\mathbb{R}^n;\mathbb{R})$$

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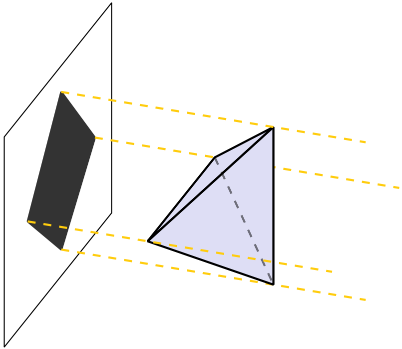
Ingredients of the Proof

- Description of supports of Monge–Ampère measures (Colesanti, Hug, JLMS '05)
- New Cauchy–Kubota formulas for mixed Monge–Ampère measures

A Classical Result

Recall: Cauchy's Surface Area Formula

$$S(K) = \frac{1}{\kappa_{n-1}} \int_{\mathbb{S}^{n-1}} \operatorname{vol}_{n-1}(\operatorname{proj}_{e^\perp} K)\, \mathrm{d}e \qquad K \in \mathcal{K}^n$$



Cauchy–Kubota Formulas for Intrinsic Volumes

Recall: Cauchy–Kubota Formulas

If $0 \leq j \leq n-1$, then

$$V_j(K) = \frac{\kappa_n}{\kappa_j \kappa_{n-j}} \binom{n}{j} \int_{\mathrm{G}(n,j)} V_j(\operatorname{proj}_E K) \,\mathrm{d}E$$

for every $K \in \mathcal{K}^n$.

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$\mathrm{G}(n,j)$ ... Grassmannian of j-dimensional linear subspaces of $\mathbb{R}^n$

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Proof

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- $K \mapsto \int_{\mathrm{G}(n,j)} V_j(\operatorname{proj}_E K) \, \mathrm{d}E$ defines a j-homogeneous, continuous, translation and rotation invariant valuation.
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Theorem (Colesanti, Ludwig, M., Amer. J. Math. '23+)

If $1 \le j \le n-1$ *and* $\alpha \in C_c([0,\infty))$, *then*

$$\overline{\mathrm{V}}^*_{j,\alpha}(v) = \frac{\kappa_n}{\kappa_j \kappa_{n-j}} \binom{n}{j} \int_{\mathrm{G}(n,j)} \overline{\mathrm{V}}^{E,*}_{j,\alpha}(v|_E)\,\mathrm{d}E$$

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For $m \in \mathbb{N}$, $K_1, \ldots, K_m \in \mathcal{K}^n$, $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\operatorname{vol}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \ldots, K_{i_n}).$$

Proof

$$\int_{\mathbb{S}^{n-1}} V_{n-1}(\operatorname{proj}_{e^\perp} K)\,\mathrm{d}e \cong \int_{\mathbb{S}^{n-1}} V(K[n-1],[0,e])\,\mathrm{d}e \cong V(K[n-1],B^n)$$

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Lemma (Hug, M., Ulivelli, '23+)

If $e \in \mathbb{S}^{n-1}$, then

$$n\mathrm{MA}(v[n-1], h_{[0,e]}; B) = \mathrm{MA}_E(v|_E; B \cap E)$$

for every $v \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ and every Borel set $B \subset \mathbb{R}^n$, where $E = e^{\perp}$.

Cauchy–Kubota Formulas for Functional Intrinsic Volumes

Theorem (Hug, M., Ulivelli, '23+)

If $1 \leq j \leq n-1$ *and* $\beta \in C_c(\mathbb{R}^n)$, *then*

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Space of Convex Conjugates

Recall: Legendre–Fenchel Transform or Convex Conjugate

$$v^*(x) = \sup_{y\in\mathbb{R}^n} \big(\langle x, y\rangle - v(y)\big), \qquad v\colon \mathbb{R}^n \to [-\infty, \infty]$$

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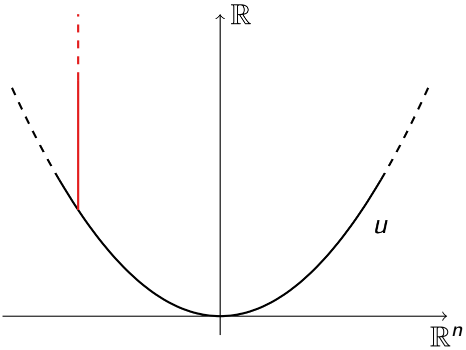

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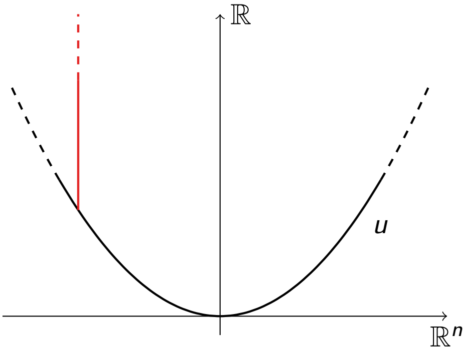

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Recall: (Effective) Domain

$$\mathrm{dom}\, u = \{x \in \mathbb{R}^n : u(x) < \infty\}$$

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Example

$$\overline{\mathrm{V}}_{n,\alpha}(\mathrm{I}_K^\infty) = \alpha(0)\,V_n(K), \quad K \in \mathcal{K}^n$$

Epi-Addition

Recall: Inf-Convolution (or Epi-Addition)

For $u_1, u_2 : \mathbb{R}^n \to (-\infty, \infty]$ convex, l.s.c.:

$$(u_1 \,\square\, u_2)(x) = \inf_{x_1+x_2=x} \big(u_1(x_1) + u_2(x_2)\big)$$

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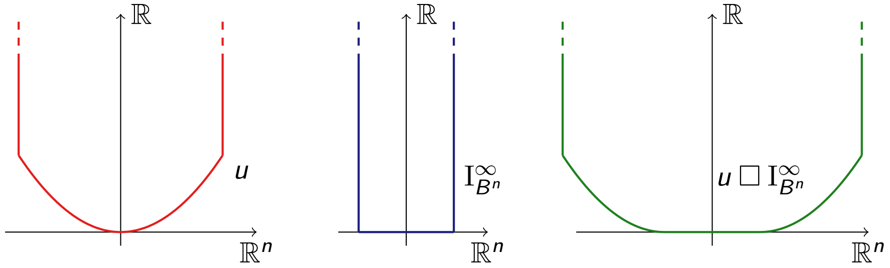

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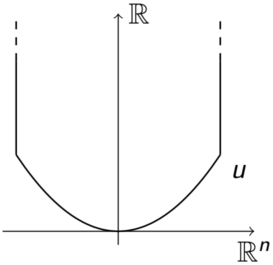
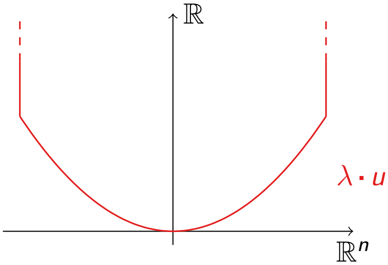
Epi-Multiplication

Epi-Multiplication on $\mathrm{Conv}_{sc}(\mathbb{R}^n)$

For $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$ and $\lambda > 0$

$$(\lambda \cdot u)(x) = \lambda u\left(\frac{x}{\lambda}\right)$$
$$\mathrm{epi}\,(\lambda \cdot u) = \lambda\, \mathrm{epi}\, u$$
$$0 \cdot u = \mathrm{I}^{\infty}_{\{o\}}$$





Functional Intrinsic Volumes on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$

Steiner Formula

$$\overline{\mathrm{V}}_{n,\alpha}(u \,\Box\, (r \cdot \mathrm{I}^{\infty}_{B^n})) = \sum_{j=0}^{n} r^{n-j} \kappa_{n-j} \overline{\mathrm{V}}_{j,\alpha}(u), \quad r > 0, u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n), \alpha \in C_c([0,\infty))$$

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The Measures $\mathrm{MA}_j^*(u;\cdot)$

Theorem (Colesanti, Ludwig, M., CVPDE '22)

For $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n) \cap C_+^2(\mathbb{R}^n)$ *and* $1 \le j \le n-1$,

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Intrinsic Volumes on Bodies of Class C^2_+

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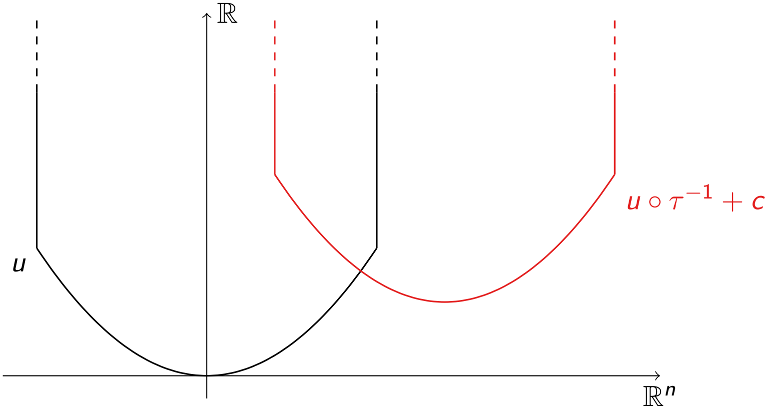

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Main Results on $\mathrm{Conv}_{sc}(\mathbb{R}^n)$

Conjugate Functional Fact of Life (Hug, M., Ulivelli, '23+)

Let $1 \leq i \leq j \leq n$ and let $\alpha, \beta \in C_c([0,\infty))$ be non-negative with $\operatorname{supp} \beta \subseteq \operatorname{supp} \alpha$. If $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$ is such that

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Local Conjugate Functional Fact of Life (Hug, M., Ulivelli, '23+)

$$\operatorname{supp} \mathrm{MA}_j^*(u,\cdot) \subseteq \operatorname{supp} \mathrm{MA}_i^*(u,\cdot), \quad 1 \le i \le j \le n, u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$$

Cauchy–Kubota Formulas on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$

Theorem (Hug, M., Ulivelli, '23+)

If $1 \leq j \leq n-1$ *and* $\beta \in C_c(\mathbb{R}^n)$, *then*

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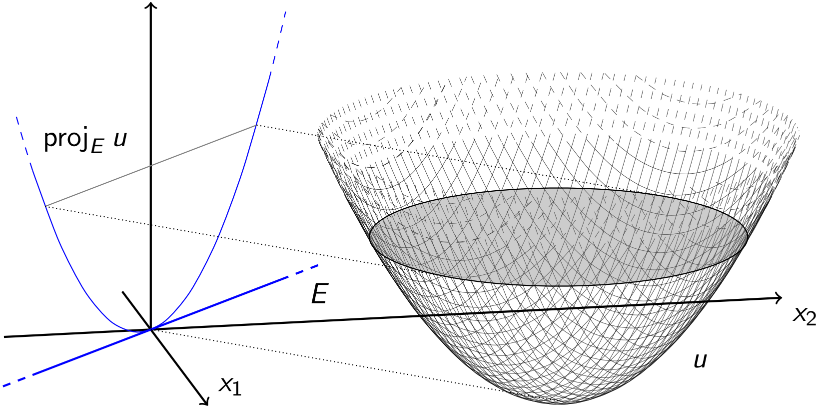

$$\mathrm{proj}_E\, u(x_E) := \min_{z\in E^\perp} u(x_E + z)$$

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Corollary

$$\int_{\mathbb{R}^n} \beta(x)\,\mathrm{dMA}_j^*(u;x) = \frac{\kappa_n}{\kappa_j}\int_{\mathrm{G}(n,j)}\int_{\mathrm{dom}\,(\mathrm{proj}_E\, u)} \beta|_E(\nabla_E\, \mathrm{proj}_E\, u(x_E))\,\mathrm{d}x_E\,\mathrm{d}E$$

From Functions to Bodies

Fix $\beta \in C_c(\mathbb{R}^n)$.

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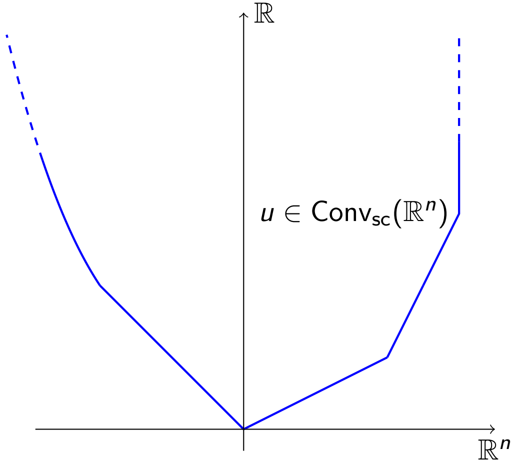

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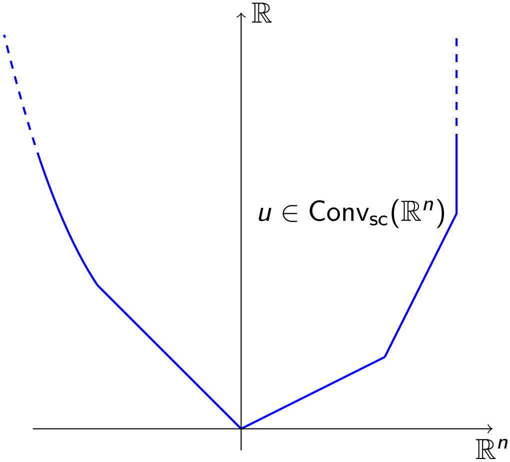

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$$\int_{\mathbb{R}^n} \beta(x)\, \mathrm{dMA}^*(u;x)$$

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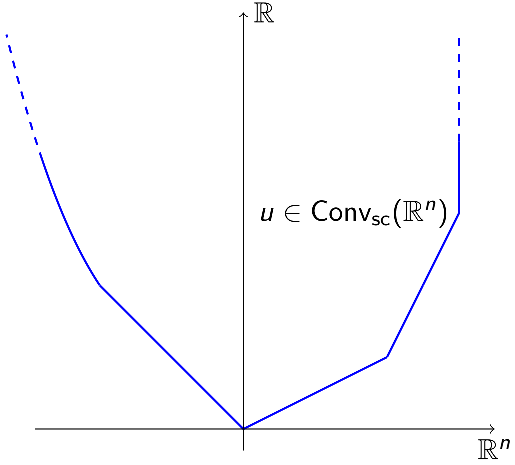

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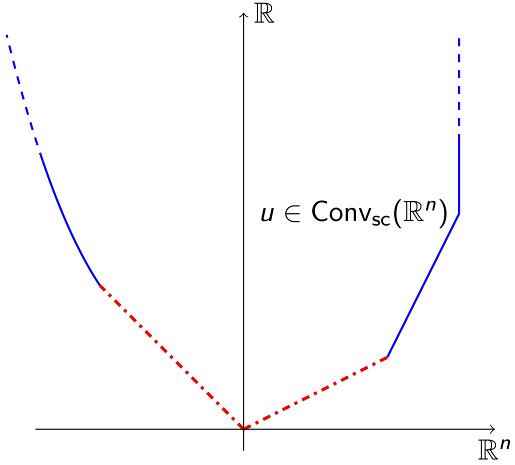

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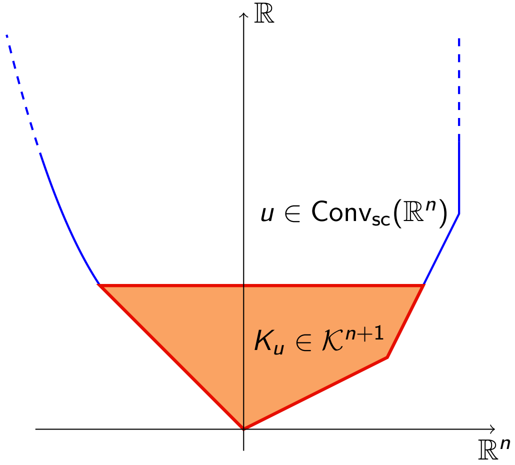

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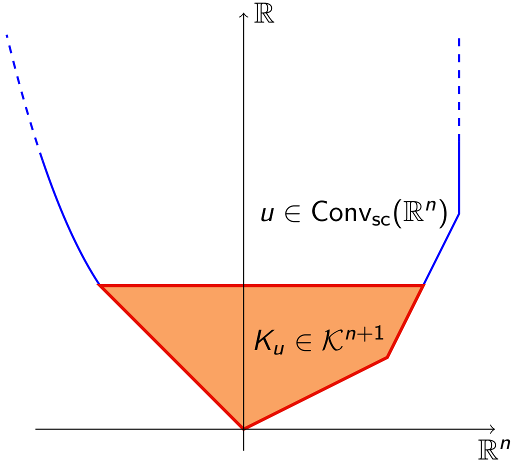

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$$\begin{aligned}\int_{\mathbb{R}^n} \beta(x)\,\mathrm{dMA}^*(u;x) &= \int_{\mathrm{dom}\, u} \beta(\nabla u(x))\,\mathrm{d}x \\ &= \int_{\mathbb{S}^n_-} \tilde{\beta}(z)\,\mathrm{d}S_n(K_u, z)\end{aligned}$$

where

$$\tilde{\beta}(z) = \frac{\beta(\mathrm{gno}(z))}{\sqrt{1+|\,\mathrm{gno}(z)|^2}}$$

for $z \in \mathbb{S}^n_-$.

Connecting MA_j^* with Special Mixed Area Measures

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$$\operatorname{supp} S(K[j], D^n[n-j], \cdot)\Big|_{\mathbb{S}^n_-} \subseteq \operatorname{supp} S(K[i], D^n[n-i], \cdot)\Big|_{\mathbb{S}^n_-}, \quad 1 \leq i \leq j \leq n, K \in \mathcal{K}^{n+1}$$

New Cauchy–Kubota Type Formulas

Theorem (Hug, M., Ulivelli, '23+)

If $1 \leq j \leq n$ *and* $\gamma \in C_c(\mathbb{S}^n_-)$, *then*

$$\frac{1}{\kappa_n}\int_{\mathbb{S}^n_-}\gamma(z)\,\mathrm{d}S(K[j],D^n[n-j],z)$$
$$=\frac{1}{\kappa_j}\int_{\substack{E\in \mathrm{G}(n+1,j)\\ \text{s.t. } e_{n+1}\in E}}\int_{\mathbb{S}^n_-\cap E}\gamma\big|_{\mathbb{S}^n_-\cap E}(z_E)\,\mathrm{d}S_{j-1}(\mathrm{proj}_E K;z_E)\,\mathrm{d}E$$

for every $K \in \mathcal{K}^{n+1}$.

The Role of Functional Intrinsic Volumes

Representation of Functional Intrinsic Volumes (Hug, M., Ulivelli, '23+)

$$\overline{\mathrm{V}}_{j,\alpha}(u) = \int_{\mathbb{S}^n_-} \tilde{\alpha}(|\langle e_{n+1}, z\rangle|) \,\mathrm{d}S(K_u[j], D^n[n-j], z)$$

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Answer (Prof. Schneider via Email)

Yes, for ω open.

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