Projection bodies in Spherical and Hyperbolic spaces

Reporter: Youjiang Lin
(Chongqing Technology and Business University
and University of Florence)
Joint work with A. Cianchi and Y. Wu

CONVEX GEOMETRY - ANALYTIC ASPECTS
Cortona, Italian
June Monday 26th - June Friday 30th 2023

1. Basic facts from Euclidean convex geometry

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Basic facts from Euclidean convex geometry

For $K \in \mathcal{K}(\mathbb{R}^n)$, K is uniquely determined by its support function h_K defined by

$$h_K(x) := \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n.$$

The support function is homogeneous of degree 1, i.e.,

$$h_K(rx) = rh_K(x), \ \text{for } r > 0. \tag{1}$$

For $K \in \mathcal{K}_o(\mathbb{R}^n)$, its radial function is defined by

$$\rho_K(x) := \max\{r > 0 : \ rx \in K\}, \quad x \in \mathbb{R}^n \backslash \{o\}. \tag{2}$$

The radial function is homogeneous of degree -1, i.e.,

$$\rho_K(rx) = \frac{1}{r}\rho_K(x), \ \text{for } r > 0. \tag{3}$$

Basic facts from Euclidean convex geometry

For $K \in \mathcal{K}_o(\mathbb{R}^n)$, its polar body is defined by

$$K^* := \{y \in \mathbb{R}^n : \ y \cdot x \le 1 \ \text{ for any } x \in K\}.$$

It is well-known that

$$(K^*)^* = K. \tag{4}$$

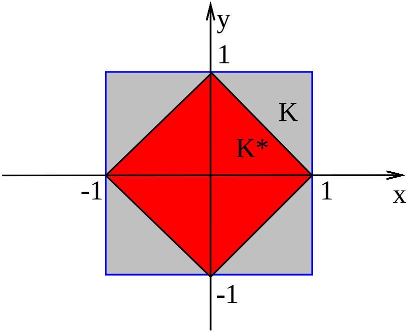


Figure. The square and its polar body

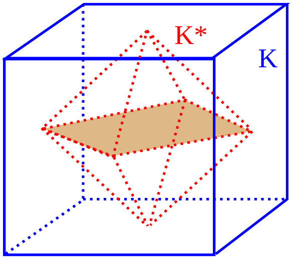


Figure. The polar body of the cube

In the plane, a polygon K has the same number of sides as its polar body K^*. And the straight line OA passing through the origin and vertex A is perpendicular to the edge corresponding to K^*, and $|OA||OB| = 1$.

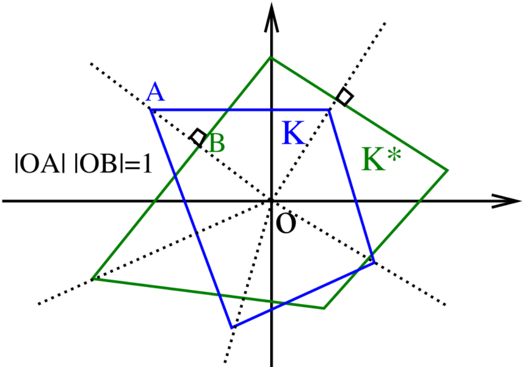
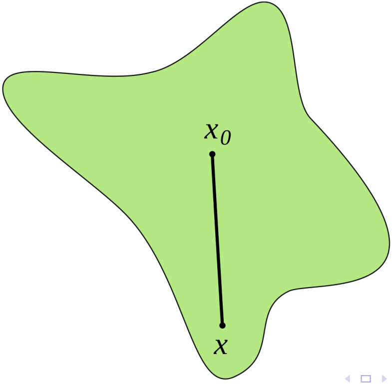


Figure. Convex polygon and its polar body in plan

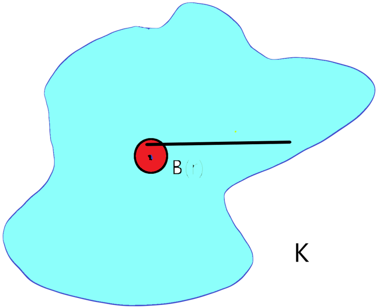
A compact set $K \subset \mathbb{R}^n$ is a *star-shaped set* with respect to the $x_0 \in K$ if the intersection of every straight line through x_0 with K is a line segment. The radial function $\rho_{K,x_0}(\cdot) : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ is defined by

$$\rho_{K,z}(x) := \max\{r \geq 0 : \ x_0 + rx \in K\}. \tag{5}$$

If ρ_{K,x_0} is strictly positive and continuous, then we call K a star body with respect to the x_0, denotes the class of star bodies in $\mathbb{R}^n$ by $\mathcal{S}_{x_0}(\mathbb{R}^n)$.



If $K \subset \mathbb{R}^n$ is a star body with respect to each point of ball $B_o(r)$, then we say K is a star body with respect to a ball. The class of star bodies with respect to ball $B_o(r)$ will be denoted by $\mathcal{S}_B(\mathbb{R}^n)$. It is clear that $\mathcal{K}_o(\mathbb{R}^n) \subset \mathcal{S}_B(\mathbb{R}^n)$, i.e., any convex body with the origin as its interior is a star body with respect to a ball.



Star body about B(r)

For $K \in \mathcal{S}_B(\mathbb{R}^n)$, its Petty projection body, denoted by $\Pi(K)$, is defined with its support function:

$$h_{\Pi(K)}(z) := \frac{1}{2} \int_{\partial K} \left| \nu^K(x) \cdot z \right| d\mathcal{H}^{n-1}(x), \tag{6}$$

where ∂K denotes the boundary of K, $\nu^K(x)$ denotes the unit outer normal vector of K at the boundary point $x \in \partial K$, "$\cdot$" denotes the Euclidean scalar product and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-Hausdorff measure. The polar body of $\Pi(K)$ will be denoted by $\Pi^*(K)$ rather than $(\Pi(K))^*$.

Basic facts from Spherical convex geometry

Let $\mathbb{R}^{n+1}$ denote $(n+1)$-dimensional Euclidean space. We denote the Euclidean unit sphere in $\mathbb{R}^{n+1}$ by $\mathbb{S}^n$, $n \geq 2$. A set $K \subseteq \mathbb{S}^n$ is called spherical convex if its radial extension

$$\operatorname{rad} K = \left\{ rv \in \mathbb{R}^{n+1} : r \geq 0 \text{ and } v \in K \right\}$$

is convex in $\mathbb{R}^{n+1}$. A closed convex subset of $\mathbb{S}^n$ is called a spherical convex body. The set of convex bodies is denoted by $\mathcal{K}(\mathbb{S}^n)$.

Basic facts from Spherical convex geometry

The spherical distance d_s is given by $d_s(u, v) = \arccos(u \cdot v)$ for $u, v \in \mathbb{S}^n$. For spherical compact sets $K, L \subset \mathbb{S}^n$, the spherical Hausdorff distance of K and L is defined by

$$d_s(K, L) := \inf\{r > 0 : \; K \subseteq L_r \; \text{and} \; L \subseteq K_r\}, \tag{7}$$

where L_r denotes the spherical parallel set of L, which is defined by

$$L_r := \{w \in \mathbb{S}^n : \; \text{there exists } v \in L \text{ such that } d_s(w, v) \leq r\}.$$

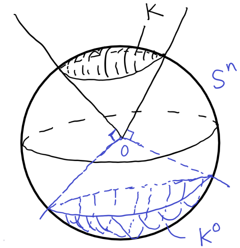
Basic facts from Spherical convex geometry

For $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, its spherical polar body K° is defined by

$$K^\circ = \{v \in \mathbb{S}^n : \ v \cdot x \leq 0 \ \text{ for all } x \in K\}. \tag{8}$$

For $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, we have

$$(K^\circ)^\circ = K. \tag{9}$$



Sphere polar body

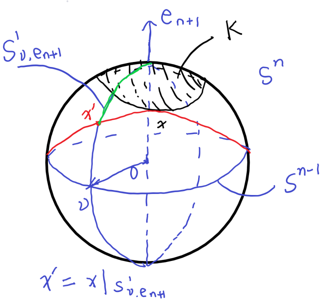
Basic facts from Spherical convex geometry

For $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, the spherical support function $h_s(K,\cdot): \mathbb{S}^{n-1} \to (0, \frac{\pi}{2})$ of K is defined by

$$h_s(K,v) = \max\left\{\operatorname{sgn}(v\cdot x)d_s\left(e_{n+1}, x \mid \mathbb{S}^1_{e_{n+1},v}\right) : x \in K\right\}, \quad v \in \mathbb{S}^{n-1}, (10$$

where $\mathbb{S}^1_{e_{n+1},v}$ denotes the 1-sphere spanned by e_{n+1} and v, and

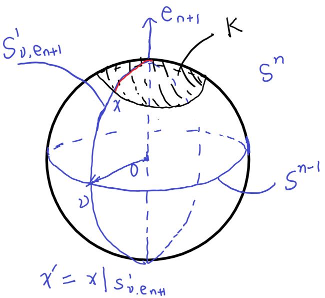
$$x|\mathbb{S}^1_{e_{n+1},v} = \mathbb{S}^1_{e_{n+1},v} \cap \operatorname{conv}\left(\left(\mathbb{S}^1_{e_{n+1},v}\right)^\circ, x\right).$$



Basic facts from Spherical convex geometry

For $K \in \mathcal{K}_o(\mathbb{S}^n_+)$, its spherical radial function is defined by

$$\rho_s(K, v) := \max\left\{\operatorname{sgn}(v \cdot x) d_s\left(e_{n+1}, x\right) : x \in K \cap \mathbb{S}^1_{e_{n+1}, v}\right\}, \quad v \in \mathbb{S}^{n-1}. (1$$

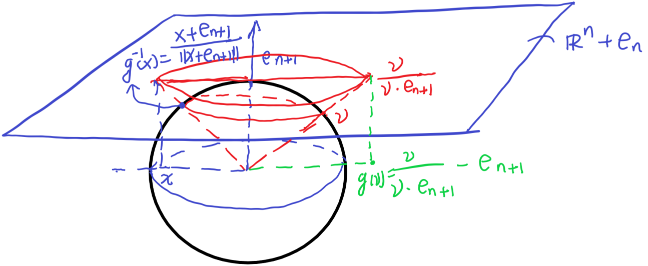


Basic facts from Spherical convex geometry

The gnomonic projection $g : \mathbb{S}^n_+ \to \mathbb{R}^n$ and the inverse gnomonic projection $g^{-1} : \mathbb{R}^n \to \mathbb{S}^n_+$ are defined by

$$g(v) := \frac{v}{e_{n+1} \cdot v} - e_{n+1} \quad and \quad g^{-1}(x) := \frac{x + e_{n+1}}{\|x + e_{n+1}\|},$$

respectively.



Basic facts from Spherical convex geometry

For a spherical compact set $K \subset \mathbb{S}^n_+$, if its gnomonic projection $g(K)$ is a star body with respect to o in $\mathbb{R}^n$, then K is called as spherical star body with respect to e_{n+1}. If $g(K)$ is a star body with respect to a ball B_o in $\mathbb{R}^n$, then K is called as spherical star body with respect to a spherical cap B_s. The set of spherical star bodies with respect to e_{n+1} is denoted by $\mathcal{S}_o(\mathbb{S}^n_+)$. The set of spherical star bodies with respect to B_s is denoted by $\mathcal{S}_B(\mathbb{S}^n_+)$.

Spherical Steiner symmetrization

For $K \in \mathcal{S}_o(\mathbb{S}^n_+)$ and $v \in \mathbb{S}^{n-1}$, its spherical Steiner symmetrization is defined by

$$\hat{S}_v(K) := g^{-1}\left(r_K S_u g(K)\right),$$

where $r_K \in (0, 1]$ such that

$$\mathcal{H}^n\left(\hat{S}_v(K)\right) = \mathcal{H}^n(K).$$

The spherical Steiner symmetrization has the following properties:

(i) If $K \in \mathcal{K}_o(\mathbb{S}^n)$, then $\hat{S}_v K \in \mathcal{K}_o(\mathbb{S}^n)$.
(ii) If $K \in \mathcal{S}_o(\mathbb{S}^n)$, then $\hat{S}_v K \in \mathcal{S}_o(\mathbb{S}^n)$.
(iii) If $K \in \mathcal{S}_B(\mathbb{S}^n)$, then $\hat{S}_v K \in \mathcal{S}_B(\mathbb{S}^n)$.

Spherical Steiner symmetrization

For $K \in \mathcal{S}_o(\mathbb{S}^n_+)$, there exists a sequence of directions $\{u_i\}_{i=1}^{\infty} \subset \mathbb{S}^{n-1}$ such that the sequence of successive spherical Steiner symmetrizations of K converges to $K^\star$ in spherical Hausdorff distance, i.e.,

$$\lim_{i\to\infty} d_s(\hat{S}_{u_i}\cdots\hat{S}_{u_1}(K), K^\star) = 0. \tag{12}$$

Here $K^\star$ is the spherical symmetric rearrangement defined as following

$$K^\star := \{v \in \mathbb{S}^n : \ d_s(v, e_{n+1}) \le \alpha, \ \ \mathcal{H}^n(K) = \mathcal{H}^n(B_s(\alpha))\}. \tag{13}$$

Spherical projection body

For $K \in \mathcal{S}_B(\mathbb{S}^n_+)$, its spherical projection body $\Pi_{\mathbb{S}}(K)$ is defined by

$$\Pi_{\mathbb{S}} K := g^{-1}\left(\Pi g(K)\right). \tag{14}$$

By the definition of spherical projection body, for $u \in \mathbb{S}^{n-1}$,

$$\tan h\left(\Pi_{\mathbb{S}} K, u\right) = \frac{1}{2} \int_{\partial g(K)} \left| u \cdot \nu^{g(K)}(y) \right| d\mathcal{H}^{n-1}(y). \tag{15}$$

Spherical projection body

The following lemma shows that the spherical projection operator $\Pi_{\mathbb{S}}: \mathcal{S}_B(\mathbb{S}^n_+) \to \mathcal{K}_o(\mathbb{S}^n)$ is continuous.

Lemma

For a sequence of spherical star bodies $\{K_i\}_{i=0}^{\infty} \subset \mathcal{S}_B(\mathbb{S}^n_+)$, if

$$\lim_{i\to\infty} d_s(K_i, K_0) = 0, \tag{16}$$

then

$$\lim_{i\to\infty} d_s(\Pi_{\mathbb{S}} K_i, \Pi_{\mathbb{S}} K_0) = 0. \tag{17}$$

Spherical projection body

Let $\bar{\mathrm{O}}(n+1)$ denote the set of rotation transformations around the x_{n+1}-axis in $\mathbb{R}^{n+1}$. The following lemma shows the rotation invariance of the spherical projection operator.

Lemma

Let $\phi \in \bar{\mathrm{O}}(n+1)$ be a rotation transformation on $\mathbb{R}^{n+1}$ and $K \in \mathcal{S}_B(\mathbb{S}^n)$. Then

$$\Pi_{\mathbb{S}}(\phi K) = \phi \Pi_{\mathbb{S}} K. \tag{18}$$

Spherical projection inequality

Theorem

If $K \in \mathcal{S}_B(\mathbb{S}^n_+)$ and $K^\star$ is the spherical cap center at e_{n+1} with the same measure as K, then

$$\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K)\right) \leq \mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}(K^\star)\right), \tag{19}$$

with the equality if and only if $K = K^\star$.

Spherical projection inequality

First, we prove the following monotonicity on the spherical Steiner symmetrization.

Lemma

Let $K \in \mathcal{S}_B(\mathbb{S}^n_+)$. Then

$$\mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}\left(\hat{S}K\right)\right) \geq \mathcal{H}^n\left(\Pi^\circ_{\mathbb{S}}K\right), \tag{20}$$

with equality if and only if $\hat{S}K = K$.

Then we use the convergence of successive spherical Steiner symmetrizations, continuity of spherical projection operator and polar operator and the above monotonicity, we can prove the main theorem.

Thank You!