The Legendre transform and dually epi-translation contravariant valuations

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Outline

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- $(\Gamma, \vee, \wedge)$ a subset of a lattice
  $(\mathbb{A}, +)$ an abelian semigroup

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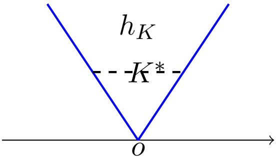
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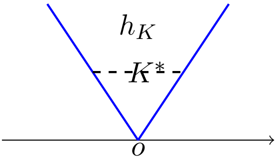
K^*
o



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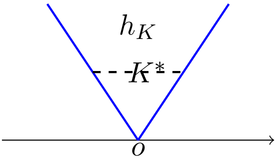

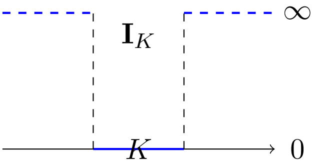

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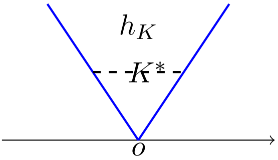

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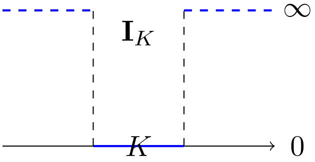

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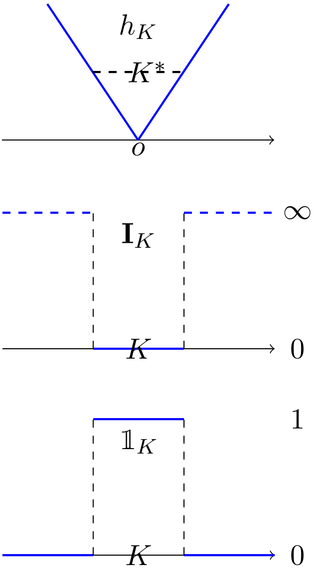

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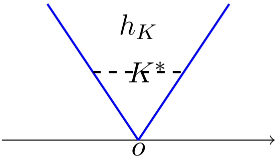

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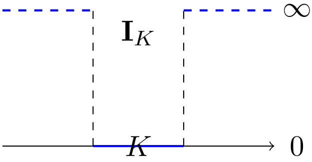

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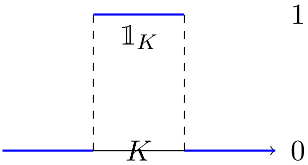

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Theorem (Artstein-Avidan & Milman: Ann. Math. 2009)

A bijective $\mathcal{Z} : \mathrm{Conv}(\mathbb{R}^n) \to \mathrm{Conv}(\mathbb{R}^n)$ *satisfying*

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- ◇ since bijection, $\mathcal{Z}$ satisfying (1) is a "stronger" valuation, i.e.,

$$\mathcal{Z}(u \vee v) = (\mathcal{Z}u) \wedge (\mathcal{Z}v),\ \mathcal{Z}(u \wedge v) = (\mathcal{Z}u) \vee (\mathcal{Z}v)$$

if $\mathrm{epi}\, u \cup \mathrm{epi}\, v$ is convex.

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$\iff$ *there exist* $c>0$ *and* $\lambda\in\mathbb{R}$ *such that*

$$\mathcal{Z}u=\frac{1}{c}u^*\circ\lambda$$

for every $u\in\mathrm{Conv}(\mathbb{R}^n)$, *and*

$$\oplus=\Box.$$

- ◇ $\mathrm{epi}(u\Box v)=\mathrm{epi}\,u+\mathrm{epi}\,v$
- ◇ $u\Box v(x):=\inf_{x=x_1+x_2}u(x_1)+v(x_2)$.

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◇ $[K, L]$: convex hull of K, L

Theorem (Böröczky & Schneider: GAFA 2008)

Let $n \geq 2$. *A transform* $\mathcal{Z} : \mathcal{K}^n_{(o)} \to \mathcal{K}^n_{(o)}$ *satisfies*

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$\iff$ *there are* $c_1, c_2 \geq 0$ *such that*

$$\mathcal{Z} K = c_1 K^* + c_2(-K^*)$$

for every $K \in \mathcal{K}^n_{(o)}$. *Here* $+$ *can be the Minkowski addition or the radial addition, and* $\mathcal{K}^n_{(o)}$ *is the set of convex bodies containing the origin.*

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- No. if no further assumptions

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◇ $\mathcal{Z} : \mathrm{Conv}_{sc}(\mathbb{R}^n) \to F(\mathbb{R}^n; \mathbb{R})$ is $\mathrm{SL}(n)$ (or $\mathrm{GL}_+(n)$) contravariant:

$$\mathcal{Z}(u \circ \phi^{-1}) = \mathcal{Z}(u) \circ \phi^t,$$

$\forall u \in \mathrm{Conv}_{sc}(\mathbb{R}^n), \phi \in \mathrm{SL}(n)$ (or $\mathrm{GL}_+(n)$).

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- Counterexample

$$\mathcal{Z}_1(u)(x) := \int_0^\infty h(\{e^{-u} \geq t\}, x)dt,\ u \in \mathrm{Conv}_{sc}(\mathbb{R}^n).$$

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$$\mathcal{Z}u = u^* + c$$

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The Hadwiger theorem on convex functions (Colesanti, Ludwig, Mussnig: 2020+)

$\mathcal{Z} : \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \to \mathbb{R}$ is a continuous, dually epi-translation invariant and rotation invariant valuation
$\Longleftrightarrow$ there are $\zeta_0 \in D_0^n, \ldots, \zeta_n \in D_n^n$

$$\mathcal{Z}(u) = V_{0,\zeta_0}^*(u) + \cdots + V_{n,\zeta_n}^*(u)$$

for every $u \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$.

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- if $u \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R}) \cap C_+^2(\mathbb{R}^n)$, then

$$V_{j,\zeta}^*(u) = \int_{\mathbb{R}^n} \zeta(|x|) \left[\mathrm{D}^2 u(x)\right]_j \, \mathrm{d}x,$$

◇ $\mathcal{Z}:\mathrm{Conv}_{sc}(\mathbb{R}^n)\to F(\mathbb{R}^n;\mathbb{R})$ is dually epi-translation contravariant:

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Let $n\geq 2$. A transform $\mathcal{Z}:\mathrm{Conv}_{sc}(\mathbb{R}^n)\to F(\mathbb{R}^n;\mathbb{R})$ is a continuous and $\mathrm{GL}_+(n)$ contravariant valuation which is dually epi-translation contravariant,

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$\Longleftrightarrow$ *there are constants* $c, c', \sigma \in \mathbb{R}$ *such that* $\eta(t) = \epsilon\sigma t$ *for every* $t \in \mathbb{R}$ *and*

$$\mathcal{Z}u(x) = \begin{cases} \epsilon\sigma u^*(x/\epsilon) + c, & \epsilon \neq 0, \\ c' \delta_x^0 + c, & \epsilon = 0, \end{cases}$$

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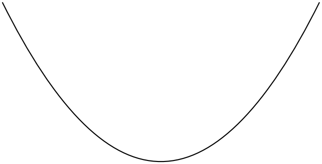
for every $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$, $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$,
if and only if there is a constant $c \in \mathbb{R}$ and $\sigma \neq 0$ such that $\eta(t) = \exp\{-\epsilon\sigma t\}$ for every $t \in \mathbb{R}$ and

$$\mathcal{Z}(u)(x) = \begin{cases} c \exp\{\epsilon\sigma u^*(x/\epsilon)\}, & \epsilon \neq 0, \\ 0, & \epsilon = 0, \end{cases}$$

for every $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Sketch of the proof



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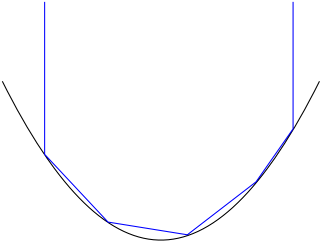

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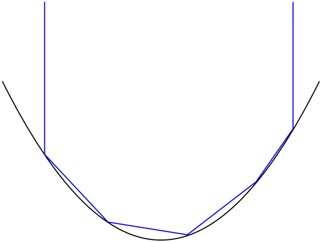

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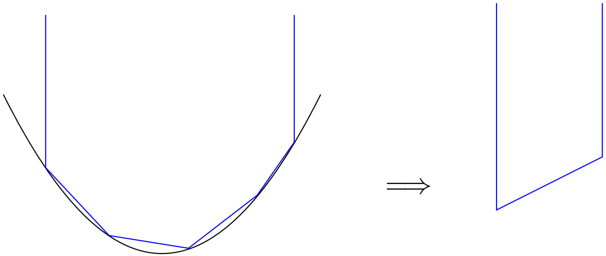

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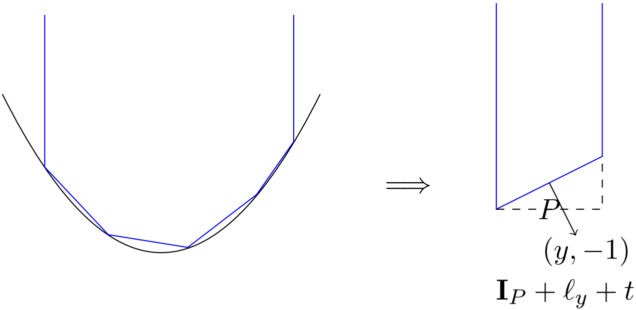

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Let $n \geq 2$. A map $Z : \mathcal{P}^n \to F(\mathbb{R}^n \setminus \{o\}; \mathbb{R})$ is a continuous and $\mathrm{GL}_+(n)$ covariant valuation

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Let $n \geq 2$. A map $Z : \mathcal{P}^n \to F(\mathbb{R}^n \setminus \{o\}; \mathbb{R})$ is a continuous and $\mathrm{GL}_+(n)$ covariant valuation
$\Longleftrightarrow$ *there are $\zeta, \tilde{\zeta} \in C(\mathbb{R})$ such that*

$$ZP(x) = \zeta(h_P(x)) + \zeta(-h_{-P}(x)) + \tilde{\zeta}(h_{[P,o]}(x)) + \tilde{\zeta}(-h_{[-P,o]}(x))$$

for every $P \in \mathcal{P}^n$ and $x \in \mathbb{R}^n \setminus \{o\}$.

◇ $\mathcal{Z}(\mathbf{I}_P - t) = \mathcal{Z}(\mathbf{I}_P) + \eta(t)$ further implies

$$\begin{aligned}\mathcal{Z}(\mathbf{I}_P + t) = \zeta(h_P(x)) + \zeta(-h_{-P}(x)) \\ + \tilde{\zeta}(h_{[P,o]}(x)) + \tilde{\zeta}(-h_{[-P,o]}(x)) - \sigma' t\end{aligned}$$

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- Together with $\mathcal{Z}(\mathbf{I}_P + \ell_{te_1}) = \tau_{\epsilon te_1}\mathcal{Z}\mathbf{I}_P$, we get

$$\begin{aligned}&\zeta(h_P(x)) + \zeta(-h_{-P}(x)) + \tilde{\zeta}(h_{[P,o]}(x)) + \tilde{\zeta}(-h_{[-P,o]}(x)) - \sigma' t \\ &= \zeta(h_P(x - \epsilon te_1)) + \zeta(-h_{-P}(x - \epsilon te_1)) \\ &\quad + \tilde{\zeta}(h_{[P,o]}(x - \epsilon te_1)) + \tilde{\zeta}(-h_{[-P,o]}(x - \epsilon te_1))\end{aligned}$$

◇ Choose good P and x to get:

$$\zeta(s) + \zeta(r) + \tilde{\zeta}(\max\{s,0\}) + \tilde{\zeta}(\min\{r,0\}) - \sigma' t$$
$$= \zeta(s-\epsilon t) + \zeta(r-\epsilon t) + \tilde{\zeta}(\max\{s-\epsilon t,0\}) + \tilde{\zeta}(\min\{r-\epsilon t,0\}).$$

for every $r, s, t \in \mathbb{R}$ with $s \geq r$.

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◇ Then

- If $\epsilon \neq 0$, then there are constants $c_1, c_2, c_3, \sigma := \sigma'/\epsilon \in \mathbb{R}$ with $c_1 - c_2 = \sigma$ such that

$$\mathcal{Z}(\mathbf{I}_P + t)(x) = c_1 h_P(x) + c_2 h_{-P}(x) + c_3 - \epsilon\sigma t$$

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    for every $P\in\mathcal{P}^n$, $x\in\mathbb{R}^n\setminus\{o\}$, and $t\in\mathbb{R}$.
  - If $\epsilon=0$, then there is a constant $c\in\mathbb{R}$ such that

$$\mathcal{Z}(\mathbf{I}_P+t)(x)=c$$

    for every $P\in\mathcal{P}^n$, $x\in\mathbb{R}^n\setminus\{o\}$, and $t\in\mathbb{R}$.

◇ For integers $i, m > 0$, consider $u_i = \mathbf{I}_{[i-1,i]\times[0,1]^{n-1}} + \ell_{ie_1} - \frac{i^2-i}{2}$ and $v_m = u_1 \wedge \cdots \wedge u_m$.

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- Let $m \to \infty$. Use $\lim_{m\to\infty} \mathcal{Z} v_m(x) = \mathcal{Z}(\lim_{m\to\infty} v_m)(x) \neq \pm\infty$ to obtain

$$\begin{cases} \text{if } \epsilon > 0, \text{ then } c_1 = \sigma, \ c_2 = 0; \\ \text{if } \epsilon < 0, \text{ then } c_2 = -\sigma, \ c_1 = 0. \end{cases}$$

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- Exactly corresponding to the desired result

$$\mathcal{Z} u(x) = \begin{cases} \epsilon\sigma u^*(x/\epsilon) + c, & \epsilon \neq 0, \\ c'\delta_x^0 + c, & \epsilon = 0. \end{cases}$$

Thank you!