Higher-Order Affine Isoperimetric Inequalities

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Kent State University

INdAM Meeting "CONVEX GEOMETRY - ANALYTIC ASPECTS"
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[1]Joint work with J. Haddad, E. Putterman, M. Roysdon, and D. Ye

Main Definitions: Mixed Volume

K and L convex bodies in $\mathbb{R}^n$ and $t \geq 0$

Then $\mathrm{Vol}_n(K + tL)$ is a homogeneous polynomial (in t) of degree n and

$$\mathrm{Vol}_n(K + tL) = \sum_{i=0}^{n} t^i \binom{n}{i} V(K[n-i], L[i]).$$

The coefficients $V(K[n-i], L[i])$ are called the mixed volumes of K $(n-i)$ times and L $[i]$ times. When $i = 1$, we write $V(K[n-1], L)$

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- Mixed volume is translation invariant: $V(K[n-1], L+a) = V(K[n-1], L)$, for $a \in \mathbb{R}^n$.
- For $T \in GL_n(\mathbb{R}^n)$: $V(TK[n-i], TL[i]) = |\det T| V(K[n-i], L[i])$. In particular: $V(K[n-1], L) = V(-K[n-1], -L)$.

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- Let B_2^n be the unit Euclidean ball in $\mathbb{R}^n$. Then: the *mean width* of K is given by

$$w_n(K) = \frac{1}{\mathrm{Vol}_n(B_2^n)} V(B_2^n[n-1], K).$$

How symmetric is a convex body?

- K is said to be centrally symmetric if $K = -K$, and to be symmetric if a translate is centrally symmetric.
- A possible candidate for a "symmetric" version of K is

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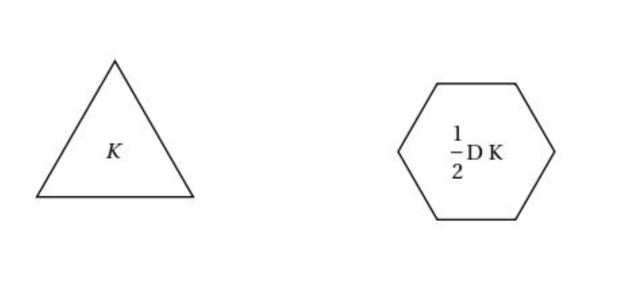

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- The **Rogers-Shephard inequality** shows the reverse direction:

$$\frac{\mathrm{Vol}_n(DK)}{\mathrm{Vol}_n(K)} \leq \binom{2n}{n},$$

  with equality if, and only if, K is a n-dimensional simplex.

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- Rolf Schneider: Define the mth order covariogram of K as

$$g_{K,m}(\bar{x}) = \mathrm{Vol}_n\left(K \cap \bigcap_{i=1}^{m} (K + x_i)\right),$$

where $\bar{x} = (x_1, \ldots, x_m) \in (\mathbb{R}^n)^m \cong \mathbb{R}^{nm}$.

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where $\bar{x} = (x_1, \ldots, x_m) \in (\mathbb{R}^n)^m \cong \mathbb{R}^{nm}$.
- The difference body of order m of K, $D^m(K)$, is a convex body in $\mathbb{R}^{nm}$ defined as the support of $g_{K,m}$.
- $$\mathrm{Vol}_n(K)^{-m}\mathrm{Vol}_{nm}(D^m(K)) \leq \binom{nm + n}{n},$$

with equality if, and only if, K is a n-dimensional simplex.

Operator Hopping

- Goal: Extend the concept of higher-order to other "symmetric" version of convex bodies

Operator Hopping

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$$\frac{d}{dr} g_K(r\theta)\big|_{r=0^+} = \frac{d}{dr}\mathrm{Vol}_n(K \cap (K + r\theta))\big|_{r=0^+} = -\mathrm{Vol}_{n-1}(P_{\theta^\perp} K),$$

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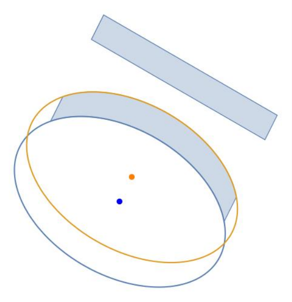

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  where $P_{\theta^\perp} K$ is the orthogonal projection of K onto the hyperplane through the origin orthogonal to θ.
- Minkowski tells us that $\mathrm{Vol}_{n-1}(P_{\theta^\perp} K) = nV(K[n-1], [o, \theta])$
- Aleksandrov tell us that $V(K[n-1], [o, \theta])$ is convex function in θ.

The Polar Projection Body

- The polar projection body of K, $\Pi^\circ K$, is the centrally symmetric convex body whose radial function is given by

$$\rho_{\Pi^\circ K}^{-1}(\theta) = nV(K[n-1],[o,\theta]).$$

- Why centrally symmetric? Translation invariance!

$$\rho_{\Pi^\circ K}^{-1}(\theta) = nV(K[n-1],[o,\theta]) = nV(K[n-1],[o,-\theta]) = \rho_{\Pi^\circ K}^{-1}(-\theta)$$

- Also, the fact that

$$\rho_{\Pi^\circ(-K)}^{-1}(\theta) = nV(-K[n-1],[o,\theta]) = nV(K[n-1],[o,-\theta]) = \rho_{\Pi^\circ K}^{-1}(-\theta)$$

shows

$$\Pi^\circ(-K) = \Pi^\circ K.$$

The Higher-order Polar Projection Body

Theorem (We Start Here)

Let K be a convex body in $\mathbb{R}^n$ and $m \in \mathbb{N}$. For every direction $\bar{\theta} = (\theta_1, \ldots, \theta_m) \in \mathbb{S}^{nm-1}$, let $C_{-\bar{\theta}} = \text{conv}_{0 \le i \le m}[o, -\theta_i]$. Then:

$$\frac{d}{dr} g_{K,m}(r\bar{\theta})\Big|_{r=0^+} = -nV(K[n-1], C_{-\bar{\theta}}).$$

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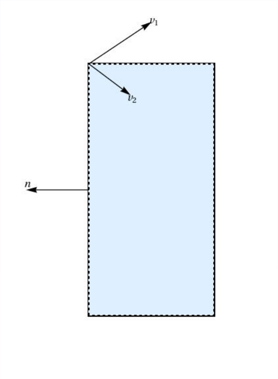

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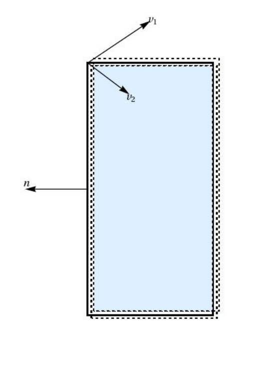

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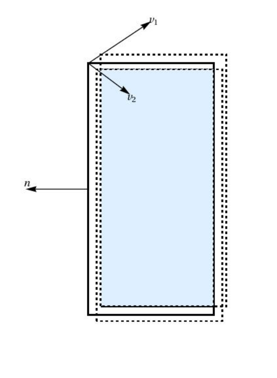

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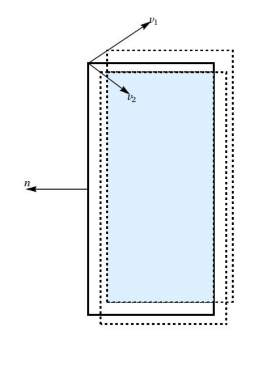

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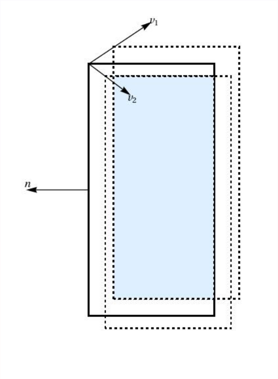

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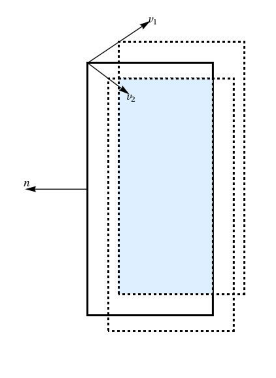

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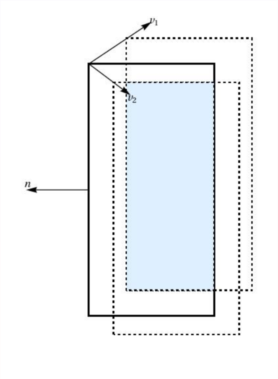

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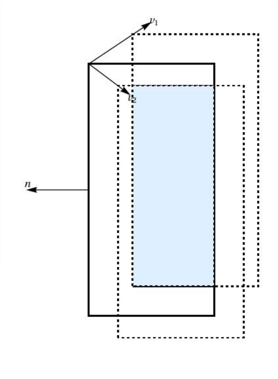

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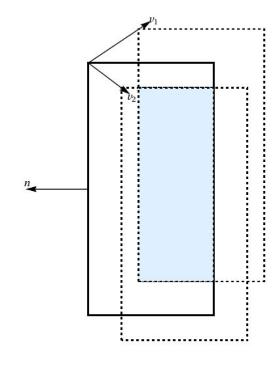

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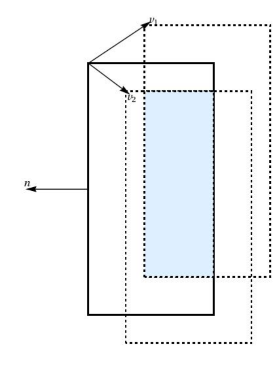

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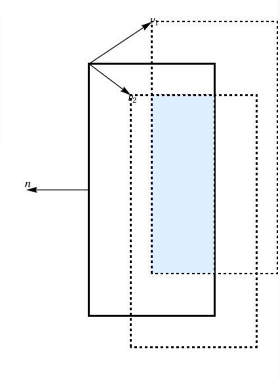

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- We define the mth order polar projection body of K as the convex body in $\mathbb{R}^{nm}$ whose radial function is given by

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- For $u \in \mathbb{S}^{n-1}$, let $u_j = (o, \dots, o, u, o, \dots, o) \in \mathbb{S}^{nm-1}$.

$$\rho_{\Pi^{\circ,m}K}(u_j)^{-1} = nV(K[n-1], [o, -u]) = \rho_{\Pi^{\circ}K}(u)^{-1}.$$

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- For $u \in \mathbb{S}^{n-1}$, let $u^m = \frac{1}{\sqrt{m}}(u, \ldots, u) \in \mathbb{S}^{nm-1}$.

$$\rho_{\Pi^{\circ,m}K}(u^m)^{-1} = \frac{n}{\sqrt{m}} V(K[n-1], [o, -u]) = \frac{1}{\sqrt{m}} \rho_{\Pi^{\circ}K}(u)^{-1}.$$

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- For $m \ge 2$, $\Pi^{\circ,m}K$ is centrally symmetric if, and only if, K is symmetric ($-\Pi^{\circ,m}K = \Pi^{\circ,m}(-K)$)

The Higher-order Polar Projection Body

Theorem (We Start Here)

Let K be a convex body in $\mathbb{R}^n$ and $m \in \mathbb{N}$. For every direction $\bar{\theta} = (\theta_1, \dots, \theta_m) \in \mathbb{S}^{nm-1}$, let $C_{-\bar{\theta}} = \text{conv}_{0 \leq i \leq m}[o, -\theta_i]$. Then:

$$\frac{d}{dr} g_{K,m}(r\bar{\theta}) \Big|_{r=0^+} = -nV(K[n-1], C_{-\bar{\theta}}).$$

- We define the mth order polar projection body of K as the convex body in $\mathbb{R}^{nm}$ whose radial function is given by

$$\rho^{-1}_{\Pi^{\circ,m}K}(\bar{\theta}) = nV(K[n-1], C_{-\bar{\theta}})$$

- $\Pi^{\circ,m}K$ contains the origin as an interior point
- For $u \in \mathbb{S}^{n-1}$, let $u_j = (o, \dots, o, u, o, \dots, o) \in \mathbb{S}^{nm-1}$.

$$\rho_{\Pi^{\circ,m}K}(u_j)^{-1} = nV(K[n-1], [o, -u]) = \rho_{\Pi^{\circ}K}(u)^{-1}.$$

- For $m \geq 2$, $\Pi^{\circ,m}K$ is centrally symmetric if, and only if, K is symmetric ($-\Pi^{\circ,m}K = \Pi^{\circ,m}(-K)$)

The Mellin Transform

Let $\psi : [0,\infty) \to [0,\infty)$ be an integrable function that is right continuous and differentiable at 0. Then, the map given by

$$\mathcal{M}_\psi : p \mapsto \begin{cases} \int_0^\infty t^{p-1}(\psi(t) - \psi(0))dt, & p \in (-1,0), \\ \int_0^\infty t^{p-1}\psi(t)dt, & p > 0 \text{ such that } t^{p-1}\psi(t) \in L^1(\mathbb{R}^+), \end{cases}$$

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For $\theta \in \mathbb{S}^{n-1}$ and a convex body K, the *radial pth mean body of K* is the compact, symmetric, star shaped set whose radial function is given by

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Note: g_K is $(1/n)$-concave. Thus, it is log-concave. Keith Ball tells us that this means R_pK is a convex body when $p \geq 0$ (0 follows by continuity).

Gardner and Zhang's Radial Mean Bodies

- Jensen's inequality tells us, for $-1 < p \leq q \leq \infty$

$$\{o\} = R_{-1}K \subset R_pK \subset R_qK \subset DK.$$

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- Berwald's inequality lets us reverse the above inclusions for $-1 < p \leq q \leq \infty$:

$$DK \subseteq \binom{n+q}{n}^{\frac{1}{q}} R_qK \subseteq \binom{n+p}{n}^{\frac{1}{p}} R_pK \subseteq n\mathrm{Vol}_n(K)\Pi^\circ K,$$

with equality if, and only if, K is a n-dimensional simplex.

Zhang's inequality

- It turns out that $\mathrm{Vol}_n(R_n K) = \mathrm{Vol}_n(K)$. Thus, the previous result implies

$$\mathrm{Vol}_n(DK) \le \binom{2n}{n}\mathrm{Vol}_n(K) \le n^n \mathrm{Vol}_n(K)^n \mathrm{Vol}_n(\Pi^\circ K).$$

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- The first inequality is the Rogers-Shephard inequality again. The second inequality is known as **Zhang's inequality**, usually written as

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Definition

For $m \in \mathbb{N}$ and $p > -1$, we define the (m,p) *radial mean bodies* $R_p^m K$, to be the star bodies (convex if $p \ge 0$) in $\mathbb{R}^{nm}$ whose radial functions are given by, for $\bar{\theta} \in \mathbb{S}^{nm-1}$:

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$$\rho_{R_p^m K}(\bar{\theta}) := \left(\frac{1}{\mathrm{Vol}_n(K)} \int_K \left(\min_{1 \leq i \leq m} \rho_{K-x}(-\theta_i) \right)^p dx \right)^{\frac{1}{p}}.$$

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$$\rho_{R_p^m K}(\bar{\theta}) = \left(p \mathcal{M}_{\frac{g_{K,m}(r\bar{\theta})}{\mathrm{Vol}_n(K)}}(p) \right)^{\frac{1}{p}} \tag{1}$$

Two Cool Technical Lemmas

Mellin-Berwald inequality by Fradelizi, Madiman and Li

For every non-increasing, s-concave, $s > 0$, function ψ, the function

$$G_\psi(p) := \left(\frac{\mathcal{M}_\psi(p)}{\mathcal{M}_{\psi_s}(p)}\right)^{1/p} = \left(p\binom{p+\frac{1}{s}}{p}\mathcal{M}_\psi(p)\right)^{1/p}$$

is decreasing on $(-1, \infty)$ (here, $\psi_s(t) = (1-t)^{1/s}$). Additionally, if there is equality for any two $p, q \in (-1, \infty)$, then $G_\psi(p)$ is constant. Furthermore, $G_\psi(p)$ is constant if, and only if, ψ^s is affine on its support.
(note: version for $s \leq 0$ also exists)

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Fractional Derivative (see e.g. Haddad and Ludwig)

If $\varphi : [0,\infty) \to [0,\infty)$ is a measurable function with $\lim_{t\to 0^+}\varphi(t) = \varphi(0)$ and such that $\int_0^\infty t^{-s_0}\varphi(t)\mathrm{d}t < \infty$ for some $s_0 \in (0,1)$, then

$$\lim_{s\to 1^-}(1-s)\int_0^\infty t^{-s}\varphi(t)\mathrm{d}t = \varphi(0).$$

Higher-Order Zhang's inequality

Theorem

Let K be a convex body in $\mathbb{R}^n$ and $m \in \mathbb{N}$. Then, for $-1 < p \leq q < \infty$, one has

$$D^m(K) \subseteq \binom{q+n}{n}^{\frac{1}{q}} R_q^m K \subseteq \binom{p+n}{n}^{\frac{1}{p}} R_p^m K \subseteq n\mathrm{Vol}_n(K)\Pi^{\circ,m}K.$$

Equality occurs in any set inclusion if, and only if, K is a n-dimensional simplex.

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- It turns out that $\mathrm{Vol}_{nm}(R_{nm}^m K) = \mathrm{Vol}_n(K)^m$.
- This fact and the above theorem yields a new proof of the higher-order Rogers-Shephard inequality.

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Zhang's inequality for higher-order projection bodies

Fix $m \in \mathbb{N}$ and K be a convex body in $\mathbb{R}^n$. Then, one has

$$\mathrm{Vol}_n(K)^{nm-m}\mathrm{Vol}_{nm}\left(\Pi^{\circ,m}K\right) \geq \frac{1}{n^{nm}}\binom{nm+n}{n},$$

with equality if, and only if, K is a n-dimensional simplex.

The Inequalities of Petty

There are two more well-known inequalities associated with $\Pi^\circ K$.

- **Petty's projection inequality**:

$$\mathrm{Vol}_n(K)^{n-1}\mathrm{Vol}_n(\Pi^\circ K) \le \left(\frac{\mathrm{Vol}_n(B_2^n)}{\mathrm{Vol}_n(B_2^{n-1})}\right)^n,$$

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- Combining the two yields the classical isoperimetric inequality

Higher-order Petty's inequalities

Theorem (Petty's projection inequality for higher-order projection bodies)

Let $m \in \mathbb{N}$ *be fixed. Then, for every convex body* K *in* $\mathbb{R}^n$*, one has*

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The proof uses a multi-dimensional Steiner symmetrization developed in two papers by (Bianchi, Gardner and Gronchi) and Ulivelli.

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The proof uses Jensen's inequality applied at the level of the orthogonal group.

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Combining both inequalities yields the isoperimetric inequality for every choice of m.

The Centroid Body

- Lutwak introduced the dual Mixed volume for star bodies K and L:

$$\widetilde{V}_i(K[n-i], L[i]) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \rho_K(\theta)^{n-i}\rho_L(\theta)^i d\theta.$$

  When $i = -1$ we write $\widetilde{V}(K[n+1], L)$.

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- Given a star body L in $\mathbb{R}^n$, its centroid body ΓL is the unique centrally symmetric convex body that satisfies the following duality: for every convex body K in $\mathbb{R}^n$, one has

$$\widetilde{V}_{-1}(L[n+1], \Pi^\circ K) = \frac{n+1}{2}\mathrm{Vol}_n(L) V(K[n-1], \Gamma L).$$

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- By setting $K = \Gamma L$ and using the so-called Dual Minkowski's inequality + Petty's projection inequality, one obtains the **Busemann-Petty centroid inequality**, which says

$$\mathrm{Vol}_n(\Gamma L)\mathrm{Vol}_n(L)^{-1}$$

  is minimized when L is a centered ellipsoid.

The Higher-Order Centroid Body

- Given a star body L in $\mathbb{R}^{nm}$, its higher-order centroid body $\Gamma^m L$ is the unique convex body in $\mathbb{R}^n$ that satisfies the following duality: for every convex body K in $\mathbb{R}^n$, one has

$$\widetilde{V}_{-1}(L[nm+1], \Pi^{\circ,m}K) = \mathrm{Vol}_{nm}(L)\frac{nm+1}{m}V(K[n-1], \Gamma^m L).$$

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- By setting $K = \Gamma^m L$ and using the so-called Dual Minkowski's inequality + the higher-order Petty's projection inequality, one obtains the Busemann-Petty centroid inequality, which says

$$\mathrm{Vol}_n(\Gamma^m L)\mathrm{Vol}_{nm}(L)^{-\frac{1}{m}}$$

is minimized when $L = \Pi^{\circ,m}E$ for an ellipsoid E.

The Random Simplex inequality

- We denote the expected volume of $C_{\bar{X}} = \mathrm{conv}_{1 \leq i \leq m}[o, X_i]$, a *random simplex of K*, by

$$\mathbb{E}_{K^n}(\mathrm{Vol}_n(C_{\bar{X}})) := \mathrm{Vol}_n(K)^{-n} \int_K \cdots \int_K \mathrm{Vol}_n(\mathrm{conv}_{1 \leq i \leq n}[o, x_i])\, dx_1 \dots dx_n.$$

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  By an observation of Petty, the right-hand side equals $2^{-n}\mathrm{Vol}_n(\Gamma K)$.
- Thus, the Busemann-Petty centroid inequality is equivalent to the **Busemann random simplex inequality**:

  $$\mathbb{E}_{K^n}(\mathrm{Vol}_n(C_{\bar{X}}))\mathrm{Vol}_n(K)^{-1} \geq \left(\frac{\mathrm{Vol}_{n-1}(B_2^{n-1})}{(n+1)\mathrm{Vol}_n(B_2^n)} \right)^n,$$

  with equality if, and only if, K is a centered ellipsoid.

The Higher order Random Simplex inequality

- Fix a convex body K in $\mathbb{R}^n$ and a star body L in $\mathbb{R}^{nm}$. Let $\bar{X} = (X_1, \ldots, X_m) \in \mathbb{R}^{nm}$ be a random vector uniformly distributed inside L, (no independence of the X_i is required).

The Higher order Random Simplex inequality

- We denote the expected mixed volume of K and $C_{\bar{X}}$ by

$$\mathbb{E}_L(V(K[n-1], C_{\bar{X}}) := \frac{1}{\mathrm{Vol}_{nm}(L)} \int_L V(K[n-1], C_{\bar{x}}) d\bar{x}.$$

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$$\mathbb{E}_L(V(K[n-1], C_{\bar{X}}) := \frac{1}{\mathrm{Vol}_{nm}(L)} \int_L V(K[n-1], C_{\bar{x}}) d\bar{x}.$$

It turns out that

$$V(K[n-1], \Gamma^m(-L)) = \mathbb{E}_L(V(K[n-1], C_{\bar{X}})).$$

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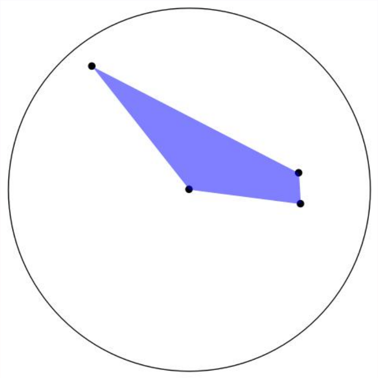

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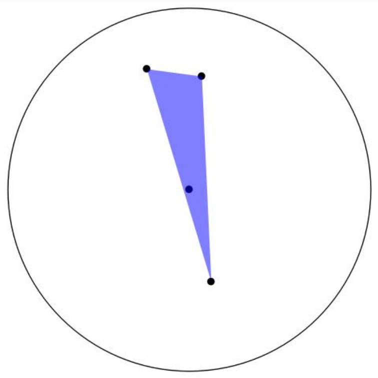

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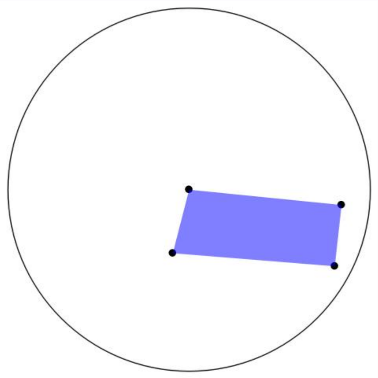

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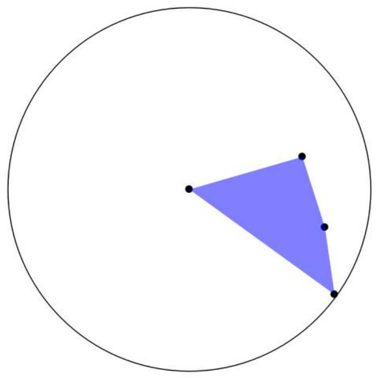

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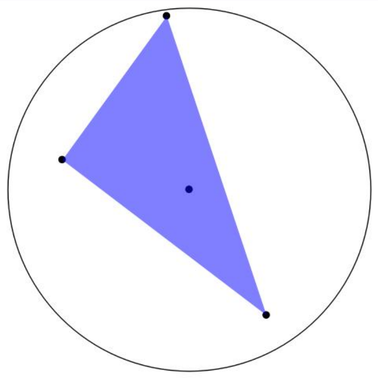

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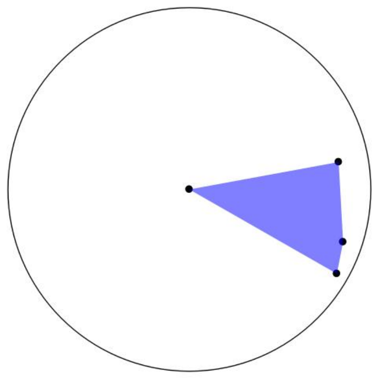

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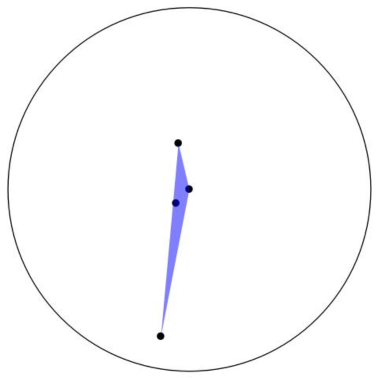

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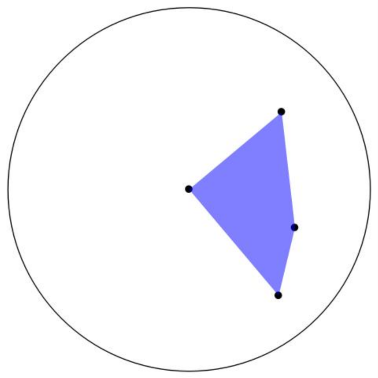

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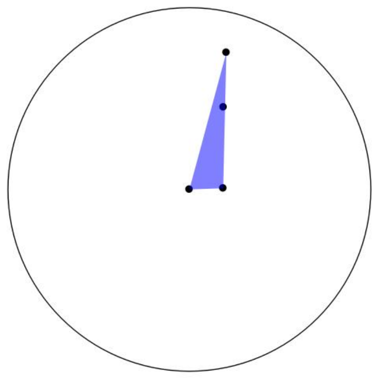

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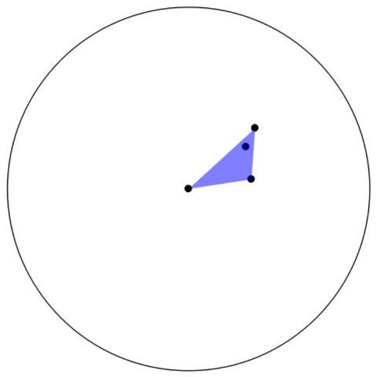

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Theorem

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It turns out that

$$\rho_{\Pi^{\circ,m} B_2^n}(\bar{x})^{-1} = n \mathrm{Vol}_n(B_2^n) w_n(C_{\bar{x}}).$$

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In fact, a special case of the above theorem is that the functional

$$\mathrm{Vol}_{nm}(L)^{-\frac{1}{nm}} \mathbb{E}_L(w_n(C_{\bar{X}})) = \mathrm{Vol}_{nm}(L)^{-\frac{nm+1}{nm}} \int_L w_n(C_{\bar{x}}) d\bar{x}$$

is minimized for $L = \lambda \Pi^{\circ,m} B_2^n$ over $\mathcal{S}^{nm}$.

BONUS: affine Sobolev's Inequality

Recall that a function f is said to be in $W^{1,1}(\mathbb{R}^n)$ if there exists a vector field ∇f satisfying

$$\int_{\mathbb{R}^n} f(x)\mathrm{div}\psi(x)dx = -\int_{\mathbb{R}^n} \langle \nabla f, \psi(x) \rangle dx$$

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This inequality can be extended to functions of bounded variation. There is equality if, and only if, there exists $A > 0$, and an ellipsoid $E \in \mathcal{K}^n$ such that $f(x) = A\chi_E(x)$.

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- The case $m = 1$ is known as Zhang's affine Sobolev inequality

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- Extends our higher-order Petty projection inequality to sets of finite perimeter

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- Implies the classical Sobolev inequality for every choice of m.