Inequalities for higher-rank mixed volumes

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Outline

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2. Alesker product
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Mixed volumes

Mixed volume

Let W be n-dimensional Euclidean space and $\mathcal{K}(W)$ the set of convex bodies.

Definition (Mixed volume)

$$V(A_1, \ldots, A_n) = \frac{1}{n!} \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \operatorname{vol}(\lambda_1 A_1 + \cdots + \lambda_n A_n)\Big|_{\lambda_1 = \cdots = \lambda_n = 0}.$$

Proposition

(a) *V is symmetric and multilinear,*

(b) $V(A, \ldots, A) = \operatorname{vol}(A)$,

(c) *The function $\phi : \mathcal{K}(W) \to \mathbb{C}$ given by*

$$\phi(A) = V\big(A[k], C_1, \ldots, C_{n-k}\big)$$

is a continuous, translation-invariant valuation:

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B).$$

Inequalities for mixed volumes

Proposition

$$V(A_1, \ldots, A_n) \geq 0.$$

Theorem (Alexandrov–Fenchel inequality)

$$V(A, B, C_1, \ldots, C_{n-2})^2 \geq V(A, A, C_1, \ldots, C_{n-2})\, V(B, B, C_1, \ldots, C_{n-2}).$$

Alesker product

Theorem (McMullen)

$$\mathrm{Val}(W) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(W).$$

Theorem (Alesker)

$$\mathrm{vol}(\cdot + A) \cdot \mathrm{vol}(\cdot + B) = \mathrm{vol}_{W \times W}\left(\Delta(\cdot) + A \times B\right)$$

defines a commutative, associative, continuous graded product on $\mathrm{Val}(W)^\infty$*.*

Remark

$$\begin{array}{ccc} \mathrm{Val}(W)^\infty \times \mathrm{Val}(W)^\infty & \xrightarrow{\text{Alesker product}} & \mathrm{Val}(W)^\infty \\ {\scriptstyle \mathbb{F}\times\mathbb{F}}\downarrow & & \downarrow{\scriptstyle \mathbb{F}} \\ \mathrm{Val}(W)^\infty \times \mathrm{Val}(W)^\infty & \xrightarrow{\text{Bernig–Fu convolution}} & \mathrm{Val}(W)^\infty \end{array}$$

Higher-rank mixed volume - the idea

Proposition (Alesker)

Let $\mathcal{A} = (A_1, \dots, A_{n-k})$ *and* $\mathcal{B} = (B_1, \dots, B_k)$*. Then*

$$V\big(\,\cdot[k], \mathcal{A}\big) \cdot V\big(\,\cdot[n-k], \mathcal{B}\big) = c_{k,n} V(\mathcal{A}, -\mathcal{B})\,\mathrm{vol}\,.$$

Let $n = k_1 + \cdots + k_\ell$ and $\mathcal{A}_i = (A_{i,1}, \dots, A_{i,n-k_i})$ for $i = 1, \dots, \ell$. Then

$$\prod_{i=1}^{\ell} V\big(\,\cdot[k_i], \mathcal{A}_i\big) = f(\mathcal{A}_1, \dots, \mathcal{A}_\ell)\,\mathrm{vol}$$

Higher-rank mixed volumes

Higher-rank mixed volume - definition

Let $n = k_1 + \cdots + k_\ell$ and $\mathcal{A}_i = (A_{i,1}, \ldots, A_{i,n-k_i})$ for $i = 1, \ldots, \ell$.

Definition (K.–Wannerer)

Let $\Delta_\ell : W \to W^\ell : x \mapsto (x, \ldots, x)$ be the diagonal embedding.

Let $\pi : W^\ell = \Delta_\ell(W) \oplus \Delta_\ell(W)^\perp \to \Delta_\ell(W)^\perp$ be the projection.

Let $\iota_i : W \to W^\ell$ be the inclusion in the i-th factor.

Let $\pi_i = \pi \circ \iota_i : W \to \Delta_\ell(W)^\perp$.

We define the **mixed volume of rank** $\ell - 1$:

$$\widetilde{V}(\mathcal{A}_1, \ldots, \mathcal{A}_\ell) := V_{\Delta_\ell(W)^\perp}\big(\pi_1(\mathcal{A}_1), \ldots, \pi_\ell(\mathcal{A}_\ell)\big),$$

where $\mathrm{vol}_{\Delta_\ell(W)} \otimes \mathrm{vol}_{\Delta_\ell(W)^\perp} = \mathrm{vol}_{W^\ell}$.

Properties of higher-rank mixed volumes

Proposition

$$\prod_{i=1}^{\ell} V\big(\,\cdot[k_i], \mathcal{A}_i\big) = c\,\widetilde{V}(\mathcal{A}_1, \ldots, \mathcal{A}_\ell)\,\mathrm{vol}\,.$$

Corollary

For $\ell = 2$ one has

$$\widetilde{V}(\mathcal{A}_1, \mathcal{A}_2) = V(\mathcal{A}_1, -\mathcal{A}_2).$$

Inequalities

Hodge–Riemann relations

Euler–Verdier involution: $(\sigma\phi)(K) = (-1)^k \phi(-K)$ for $\phi \in \mathrm{Val}_k$.

Theorem (K.-Wannerer)

Let $\mathcal{A}_1, \ldots, \mathcal{A}_{n-1}$ be $(n-1)$-tuples of centrally symmetric bodies and denote

$$\psi_i = V(\cdot, \mathcal{A}_i) \in \mathrm{Val}_1^\infty .$$

If $\phi \in \mathrm{Val}_1^\infty$ satisfies $\phi \cdot \prod_{i=1}^{n-1} \psi_i = 0$, then

$$\overline{\sigma\phi} \cdot \phi \cdot \prod_{i=1}^{n-2} \psi_i \geq 0.$$

Proof.

$$\begin{aligned}(\mathrm{AF}) &\implies \frac{V(A,A,\mathcal{C})}{V(A,K,\mathcal{C})^2} - \frac{2V(A,B,\mathcal{C})}{V(A,K,\mathcal{C})V(B,K,\mathcal{C})} + \frac{V(B,B,\mathcal{C})}{V(B,K,\mathcal{C})^2} \leq 0 \\ &\implies (\mathrm{HR}, *) \\ &\overset{\mathbb{F}}{\implies} (\mathrm{HR}, \cdot)\end{aligned}$$

Alexandrov–Fenchel-type inequality

Theorem (K.-Wannerer)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \dots, \mathcal{C}_{n-2} \in \mathcal{K}(W)^{n-1}$ *with* $\mathcal{B}, \mathcal{C}_1, \dots, \mathcal{C}_{n-2}$ *symmetric.*
Then the mixed volume of rank $n-1$ *satisfies*

$$\widetilde{V}(\mathcal{A}, -\mathcal{B}, \mathcal{C}_1, \dots, \mathcal{C}_{n-2})^2 \geq \widetilde{V}(\mathcal{A}, -\mathcal{A}, \mathcal{C}_1, \dots, \mathcal{C}_{n-2})\, \widetilde{V}(\mathcal{B}, -\mathcal{B}, \mathcal{C}_1, \dots, \mathcal{C}_{n-2}).$$

Proof.

(HR) for

$$\phi = V(\cdot, \mathcal{A}) - \frac{\widetilde{V}(\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \dots, \mathcal{C}_{n-2})}{\widetilde{V}(\mathcal{B}, \mathcal{B}, \mathcal{C}_1, \dots, \mathcal{C}_{n-2})} V(\cdot, \mathcal{B}).$$

Conjecture

The symmetry assumption (at least on $\mathcal{B}$*) is not necessary.*

Back to the (usual) mixed volume

Corollary

Let $\xi:\mathbb{R}^3 \to \mathbb{R}$ *be a linear functional and consider the graphing map*

$$\bar{\xi}:\mathbb{R}^3 \to \mathbb{R}^4, \quad \bar{\xi}(x) = \big(x, \xi(x)\big).$$

Then for all $A_1, A_2, B_1, B_2 \in \mathcal{K}(\mathbb{R}^3)$ *with* $B_i = -B_i$*, one has*

$$V\big(A_1, A_2, \bar{\xi}(B_1), \bar{\xi}(B_2)\big)^2 \geq V\big(A_1, A_2, \bar{\xi}(A_1), \bar{\xi}(A_2)\big)\, V\big(B_1, B_2, \bar{\xi}(B_1), \bar{\xi}(B_2)\big).$$

Proof.

Let $A, B, C \in \mathcal{K}(\mathbb{R}^3)$ with $C \subset \ker \xi$. Applying

$$f : (\mathbb{R}^3)^3 \to (\mathbb{R}^3)^2 : (w_1, w_2, w_3) \mapsto (w_1 - w_3, w_2 - w_3)$$

and

$$\varphi : (\mathbb{R}^3)^2 \to \mathbb{R}^3 \times \mathbb{R} : (w_1, w_2) \mapsto \big(w_1 - w_2, -\xi(w_2)\big),$$

we get

$$\begin{aligned} \widetilde{V}\big(A[2], B[2], C[2]\big) &= V_{\Delta_3(\mathbb{R}^3)^\perp}\big(\pi_1(A)[2], \pi_2(B)[2], \pi_3(C)[2]\big) \\ &= c\, V_{(\mathbb{R}^3)^2}\big(\iota_1(A)[2], \iota_2(B)[2], \Delta_2(-C)[2]\big) \\ &= \hat{c}(C)\, V_{\mathbb{R}^3\times\mathbb{R}}\big(A[2], \bar{\xi}(-B)[2]\big). \end{aligned}$$

Thank you!