Comparison problems for the Radon transform.

Alexander Koldobsky

University of Missouri-Columbia

Joint work with Michael Roysdon and Artem Zvavitch

Settings.

Given two non-negative functions f, g such that the Radon transform of f is pointwise smaller than the Radon transform of g, does it follow that the L^p-norm of f is smaller than the L^p-norm of g for a given $p > 0$?

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For a function φ on $\mathbb{R}^n$, integrable over all affine hyperplanes, the (classical) Radon transform of φ is the function $\mathcal{R}\varphi$ on $\mathbb{R} \times S^{n-1}$ defined by

$$\mathcal{R}\varphi(t,\theta) = \int_{\langle x,\theta\rangle = t} \varphi(x)dx, \quad (t,\theta) \in \mathbb{R} \times S^{n-1},$$

where integration is over the Lebesgue measure in the hyperplane perpendicular to θ at distance t from the origin.

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The spherical Radon transform of a continuous function f on the sphere S^{n-1} is a continuous function Rf on the sphere defined by

$$Rf(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi)d\xi, \quad \theta \in S^{n-1}.$$

Here $\theta^\perp$ denotes the central hyperplane orthogonal to the direction θ.

Problem 1. *Consider two even, continuous, positive functions* f, g *on* S^{n-1}, $n \geq 3$, *and let* $p > 0$. *If*

$$Rf(\theta) \leq Rg(\theta) \quad \textit{for all } \theta \in S^{n-1}, \tag{1}$$

does it follow that $\|f\|_{L^p(S^{n-1})} \leq \|g\|_{L^p(S^{n-1})}$?

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Problem 2. *Let* $p > 0$. *Given a pair of even, continuous positive functions* $\varphi, \psi \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $n \geq 2$, *satisfying the condition*

$$\mathcal{R}\varphi(t,\theta) \leq \mathcal{R}\psi(t,\theta), \quad \text{for all } (t,\theta) \in \mathbb{R} \times S^{n-1}, \tag{2}$$

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Let $n \geq 2$, $M > 1$, $p > \frac{n}{n-1}$. Then the functions $\varphi(x) = \chi_{B_2^n}(x)$ and $\psi(x) = M^{-n+1}\chi_{MB_2^n}(x)$ provide a counterexample to Problem 2.

Motivation: The Busemann-Petty problem 1.

1956: Suppose $K, L \subset \mathbb{R}^n$ are two origin-symmetric convex bodies so that

$$|K \cap \theta^\perp| \le |L \cap \theta^\perp|, \quad \forall \theta \in S^{n-1}.$$

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Indeed, by the polar formulas for volume

$$|K \cap \xi^\perp| = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|x\|_K^{-n+1} dx = \frac{1}{n-1} R(\|\cdot\|_K^{-n+1})(\xi) = \frac{1}{n-1} Rf(\xi),$$

and

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|x\|_K^{-n} dx = \frac{1}{n} \|f\|_{L^p(S^{n-1})}^p.$$

Motivation: The Busemann-Petty problem 2.

One of the ingredients of the solution of the BP problem is Lutwak's connection with intersection bodies. Lutwak showed that if the body K is an intersection body, then the answer to the Busemann-Petty problem is affirmative for any star body L. On the other hand, every origin-symmetric convex non-intersection body can be perturbed to construct a counterexample. Therefore, the answer to the Busemann-Petty problem in $\mathbb{R}^n$ is affirmative if, and only if every origin-symmetric convex body in $\mathbb{R}^n$ is an intersection body.

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Another ingredient in the Fourier analytic solution of the Busemann-Petty problem is the characterization of intersection bodies in terms of the Fourier transform. It was proved in K.(1998) that an origin-symmetric star body $K \subset \mathbb{R}^n$ is an intersection body if, and only if, $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$. Recall that a distribution f is called positive definite if $\langle \hat{f}, \phi \rangle \geq 0$ for any non-negative test function $\phi \in \mathcal{S}(\mathbb{R}^n)$.

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Our approach to the comparison problems is based on these two ideas. We introduce special classes of functions that play the role of intersection bodies. For the spherical comparison problem, this is the class of functions f on S^{n-1} for which the extension of f^{p-1} to an even homogeneous of degree -1 function on $\mathbb{R}^n$ represents a positive definite distribution. The results resemble Lutwak's connections in the Busemann-Petty problem.

Intersection bodies.

A closed bounded set K in $\mathbb{R}^n$ is called a **star body** if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the **Minkowski functional** of K defined by $\|x\|_K = \min\{a \geq 0 : \ x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

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The **radial function** of a star body K is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of K in the direction of x.

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The class of intersection bodies was introduced by Lutwak. We say that a star body K in $\mathbb{R}^n$ is the **intersection body of** another star body L if for every $\xi \in S^{n-1}$,

$$r_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp| = \frac{1}{n-1} R\left(\|\cdot\|_L^{-n+1}\right)(\xi).$$

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If μ is a finite Borel measure on S^{n-1}, then $R\mu$ is defined by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x), \qquad \forall f \in C(S^{n-1}).$$

A star body K in $\mathbb{R}^n$ is called an **intersection body** if $\|\cdot\|_K^{-1} = R\mu$ for some measure μ, i.e.

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x) dx = \int_{S^{n-1}} Rf(x) d\mu(x), \qquad \forall f \in C(S^{n-1}).$$

Theorem 1. *Let f, g be even continuous positive functions on the sphere S^{n-1}, and suppose that*

$$Rf(\theta) \leq Rg(\theta), \qquad \text{for all } \theta \in S^{n-1}. \tag{3}$$

Then:

(a) *Suppose that for some $p > 1$ the function $|x|_2^{-1} f^{p-1}\left(\frac{x}{|x|_2}\right)$ represents a positive definite distribution on $\mathbb{R}^n$. Then $\|f\|_{L^p(S^{n-1})} \leq \|g\|_{L^p(S^{n-1})}$.*

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(b) *Suppose that for some $0 < p < 1$ the function $|x|_2^{-1} g^{p-1}\left(\frac{x}{|x|_2}\right)$ represents a positive definite distribution on $\mathbb{R}^n$. Then $\|f\|_{L^p(S^{n-1})} \le \|g\|_{L^p(S^{n-1})}$.*

Theorem 2. *The following hold true:*

(a) *Let g be an infinitely smooth strictly positive even function on S^{n-1} and $p>1$. Suppose that the distribution $|x|_2^{-1} g^{p-1}\left(\frac{x}{|x|_2}\right)$ is not positive definite on $\mathbb{R}^n$. Then there exists an infinitely smooth even function f on S^{n-1} so that the condition (3) holds, but $\|f\|_{L^p(S^{n-1})} > \|g\|_{L^p(S^{n-1})}$.*

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(b) *Let f be an infinitely smooth strictly positive even function on S^{n-1} and $0 < p < 1$. Suppose that the distribution $|x|_2^{-1} f^{p-1}\left(\frac{x}{|x|_2}\right)$ is not positive definite on $\mathbb{R}^n$. Then there exists an infinitely smooth even function g on S^{n-1} so that the condition (3) holds, but $\|f\|_{L^p(S^{n-1})} > \|g\|_{L^p(S^{n-1})}$.*

Definition. Let g be a positive, continuous, integrable, and even in the first variable function on $\mathbb{R} \times S^{n-1}$. We say that a function f on $\mathbb{R}^n$ is an intersection function of g if, for any Schwartz test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\varphi(x)\ dx = \int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}\varphi(t,\theta) g(t,\theta)\ dt\ d\theta.$$

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The existence of an intersection function is guaranteed by the well-known formula for the dual Radon transform:

Proposition 1. The function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$f(x) = \int_{S^{n-1}} g(\langle x,\theta\rangle,\theta)d\theta$$

is an intersection function of g.

Proposition 2. *Let g be as above. A function f on* $\mathbb{R}^n$ *is an intersection function of g if, and only if,*

$$f = \frac{1}{\pi}\left(|x|_2^{-n+1}\left(g\left(t, \frac{x}{|x|_2}\right)\right)_t^{\wedge}(|x|_2)\right)_x^{\wedge},$$

where the interior Fourier transform is taken with respect to $t \in \mathbb{R}$*, and the exterior Fourier transform is with respect to* $x \in \mathbb{R}^n$*.*

Proof. Note that for fixed $\theta \in S^{n-1}$ the function $t \in \mathbb{R} \to \mathcal{R}\hat{\varphi}(t,\theta)$ is the Fourier transform of the function $z \in \mathbb{R} \to (2\pi)^{n-1}\varphi(z\theta)$. Therefore, for any test function φ, applying Parseval's identity to the inner integral by dt, we get

$$\langle \hat{f}, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)\, dx = \int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}\hat{\varphi}(t,\theta) g(t,\theta)\, dt\, d\theta$$

$$= (2\pi)^{n-1} \int_{S^{n-1}} \int_{\mathbb{R}} \varphi(z\theta)(g(t,\theta))_t^{\wedge}(z)\, dz\, d\theta$$

$$= 2(2\pi)^{n-1}\left\langle |x|_2^{-n+1}\left(g\left(t, \frac{x}{|x|_2}\right)\right)_t^{\wedge}(|x|_2), \varphi(x)\right\rangle. \qquad \square$$

Definition. *A function f defined on $\mathbb{R}^n$ is called an* **intersection function** *if there exists a non-negative, even, finite Borel measure μ on $\mathbb{R} \times S^{n-1}$ such that*

$$\int_{\mathbb{R}^n} f\varphi = \int_{\mathbb{R}\times S^{n-1}} \mathcal{R}\varphi(t,\theta)d\mu(t,\theta), \quad \forall\varphi \in \mathcal{S}(\mathbb{R}^n).$$

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Suppose that L is an intersection body corresponding to the measure ν on the sphere. Recall the definition of the intersection body,

$$\int_{\mathbb{R}^n} f\varphi = \int_{S^{n-1}} R\varphi(\theta)d\nu(\theta), \quad \forall \varphi \in S^{n-1}.$$

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For each θ, the function $t\mapsto |t|^{n-1}\widehat{f}(t\theta)$ must be positive definite.

We now formulate analogs of Lutwak's connections for Problem 2.

Theorem 4. *Let $p > 0$ and consider a pair of continuous, non-negative even functions $\varphi, \psi \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ satisfying the condition*

$$\mathcal{R}\varphi(t,\theta) \le \mathcal{R}\psi(t,\theta) \quad \text{for all } (t,\theta) \in \mathbb{R} \times S^{n-1}. \tag{4}$$

Then:

(a) *if $p > 1$ and φ^{p-1} is an intersection function, then*

$$\|\varphi\|_{L^p(\mathbb{R}^n)} \le \|\psi\|_{L^p(\mathbb{R}^n)};$$

(b) *if $0 < p < 1$ and ψ^{p-1} is an intersection function, then*

$$\|\varphi\|_{L^p(\mathbb{R}^n)} \le \|\psi\|_{L^p(\mathbb{R}^n)}.$$

.

We also give a counterexample to Problem 2.

Theorem 5.

(a) *Fix* $p > 1$ *and let* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *be strictly positive and even. If* ψ^{p-1} *is not an intersection function, then there exists an even, non-negative function* φ *such that* $\varphi^{p-1} \in \mathcal{S}(\mathbb{R}^n)$ *and*

$$\mathcal{R}\varphi(t,\theta) \leq \mathcal{R}\psi(t,\theta) \quad \textit{for all } (t,\theta) \in \mathbb{R} \times S^{n-1},$$

but with $\|\psi\|_{L^p(\mathbb{R}^n)} < \|\varphi\|_{L^p(\mathbb{R}^n)}$.

(b) *Fix* $0 < p < 1$ *and let* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *be strictly positive and even. If* φ^{p-1} *is not an intersection function, then there exists a non-negative, even function* ψ *such that* $\psi^{p-1} \in \mathcal{S}(\mathbb{R}^n)$ *and*

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Slicing inequalities 1.

The slicing problem of Bourgain: Does there exist an absolute constant $C > 0$ such that, for any $n \in \mathbb{N}$ and for any origin-symmetric convex body K in $\mathbb{R}^n$,

$$|K|^{\frac{n-1}{n}} \le C \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}|?$$

The best-to-date estimate $C \le O(\sqrt{\log n})$ is due to Klartag.

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A version for arbitrary functions was proved in K. 2015: for any $n \in \mathbb{N}$, any star body K in $\mathbb{R}^n$ and any non-negative continuous function f on K, one has

$$\int_K f \leq 2 d_{\mathrm{ovr}}(K, \mathcal{I}_n)\ |K|^{\frac{1}{n}} \max_{\theta \in S^{n-1}} Rf(\theta),$$

where

$$d_{\mathrm{ovr}}(K, \mathcal{I}_n) = \inf \left\{ \left(\frac{|D|}{|K|} \right)^{1/n} :\ K \subset D,\ D \in \mathcal{I}_n \right\}.$$

The slicing problem of Bourgain: Does there exist an absolute constant $C > 0$ such that, for any $n \in \mathbb{N}$ and for any origin-symmetric convex body K in $\mathbb{R}^n$,

$$|K|^{\frac{n-1}{n}} \le C \max_{\theta \in S^{n-1}} |K \cap \theta^\perp|?$$

The best-to-date estimate $C \le O(\sqrt{\log n})$ is due to Klartag.

A version for arbitrary functions was proved in K. 2015: for any $n \in \mathbb{N}$, any star body K in $\mathbb{R}^n$ and any non-negative continuous function f on K, one has

$$\int_K f \le 2 d_{\rm ovr}(K, \mathcal{I}_n)\ |K|^{\frac{1}{n}} \max_{\theta \in S^{n-1}} Rf(\theta),$$

where

$$d_{\rm ovr}(K, \mathcal{I}_n) = \inf \left\{ \left(\frac{|D|}{|K|} \right)^{1/n} :\ K \subset D,\ D \in \mathcal{I}_n \right\}.$$

This means that if K is origin-symmetric convex and $|K| = 1$, $\int_K f = 1$, then there exists a direction θ for which $Rf(\theta) \ge \frac{1}{2\sqrt{n}}$.

We get a slicing inequality of a different kind from Theorem 1. If the function g is constant with the value

$$g \equiv \frac{1}{|S^{n-2}|} \max_{\xi \in S^{n-1}} \int_{S^{n-1} \cap \xi^\perp} f(\theta) d\theta,$$

then f and g satisfy the conditions of the theorem, and we get

Corollary 1. Let f be a positive even, continuous function on the sphere S^{n-1} Assume $p > 1$ and if $|x|_2^{-1} f^{p-1}\left(\frac{x}{|x|_2}\right)$ represents a positive definite distribution on $\mathbb{R}^n$, then

$$\|f\|_{L^p(S^{n-1})} \leq \frac{|S^{n-1}|^{\frac{1}{p}}}{|S^{n-2}|} \max_{\xi \in S^{n-1}} Rf(\xi).$$

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Similarly, in the case $0 < p < 1$ we get

Corollary 2. *Let g be a positive even, continuous function on the sphere S^{n-1}. Assume that $0 < p < 1$ and $|x|_2^{-1} g^{p-1}\left(\frac{x}{|x|_2}\right)$ represents a positive definite distribution on $\mathbb{R}^n$, then*

$$\|g\|_{L^p(S^{n-1})} \geq \frac{|S^{n-1}|^{\frac{1}{p}}}{|S^{n-2}|} \min_{\xi \in S^{n-1}} Rg(\xi).$$

As proved in K.1998, an origin-symmetric star body $K \subset \mathbb{R}^n$ is an intersection body if, and only if, $\|\cdot\|_K^{-1}$ represents a positive definite distribution on $\mathbb{R}^n$. Therefore, a positive even continuous function f on the sphere has the property that the distribution $f^{p-1} \cdot r^{-1}$ is positive definite if, and only if, $f = \|\cdot\|_K^{-\frac{1}{p-1}}$ for some intersection body K.

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Combining this observation with Corollary 1, we get that for any intersection body K in $\mathbb{R}^n$ and any $p > 1$

$$\left(\int_{S^{n-1}} \|x\|_K^{-\frac{p}{p-1}} dx\right)^{\frac{1}{p}} \leq \frac{|S^{n-1}|^{\frac{1}{p}}}{|S^{n-2}|} \max_{\xi \in S^{n-1}} \left(\int_{S^{n-1} \cap \xi^\perp} \|x\|_K^{-\frac{1}{p-1}} dx\right).$$

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When $p = \frac{n}{n-1}$, the latter inequality turns into Bourgain's slicing inequality for intersection bodies.

$$\left(\int_{S^{n-1}} \|x\|_K^{-n} dx\right)^{\frac{n-1}{n}} \le \frac{|S^{n-1}|^{\frac{n-1}{n}}}{|S^{n-2}|} \max_{\xi \in S^{n-1}} \left(\int_{S^{n-1} \cap \xi^\perp} \|x\|_K^{-n+1} dx\right).$$

Connection with inequalities of the Oberlin-Stein type.

The inequality of Corollary 1 can be considered as the reverse to the Oberlin-Stein type inequalities for the Radon transform.

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Oberlin and Stein: Given any function $f \in L^p(\mathbb{R}^n)$, one has that

$$\left(\int_{S^{n-1}} \left(\int_{\mathbb{R}} |\mathcal{R}f(t,\theta)|^r dt \right)^{\frac{q}{r}} d\theta \right)^{\frac{1}{q}} \leq C_{n,p,q} \|f\|_{L^p(\mathbb{R}^n)},$$

if, and only if, $1 \leq p < \frac{n}{n-1}$, $q \leq p'$ $(p^{-1} + p'^{-1} = 1)$, and $\frac{1}{r} = \frac{n}{p} - n + 1$.

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It was also proved by Oberlin-Stein that for every $n \ge 3$ one has

$$\left(\int_{S^{n-1}} \sup_{t\in\mathbb{R}} |\mathcal{R}f(t,\theta)|^s d\theta\right)^{\frac{1}{s}} \le C_{p_1,p_2,s}\|f\|^{\alpha}_{L^{p_1}(\mathbb{R}^n)}\|f\|^{1-\alpha}_{L^{p_2}(\mathbb{R}^n)}$$

whenever $s \le n$, $1 \le p_1 < \frac{n}{n-1} < p_2 \le \infty$, and $\frac{\alpha}{p_1} + \frac{1-\alpha}{p_2} = \frac{n-1}{n}$.

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whenever $s \leq n$, $1 \leq p_1 < \frac{n}{n-1} < p_2 \leq \infty$, and $\frac{\alpha}{p_1} + \frac{1-\alpha}{p_2} = \frac{n-1}{n}$.
If χ_A is the characteristic function of a measurable set $A \subset \mathbb{R}^n$ and $s = n$, then as $p_1, p_2 \to \frac{n}{n-1}$, the latter inequality becomes

$$\left(\int_{S^{n-1}} \left(\sup_{t \in \mathbb{R}} |A \cap (\theta^{\perp} + t\theta)| \right)^n \right)^{\frac{1}{n}} \leq C_n |A|^{\frac{1}{n}}.$$

If A is an origin-symmetric convex body in $\mathbb{R}^n$, by Brunn's theorem the supremum is achieved at $t = 0$, and one gets and the Busemann intersection inequality. This connection was first observed by Lutwak.