Monge-Ampère operators and valuations

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Valuations on sets and functions

$\mathcal{S}$: family of sets, $(G, +)$: Abelian semi-group

Definition

$\mu : \mathcal{S} \to G$ is called a valuation if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

for all $K, L \in \mathcal{S}$ s.t. $K \cup L, K \cap L \in \mathcal{S}$.

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X: some family of real valued functions

Definition

$\mu : X \to G$ is called a valuation if

$$\mu(f \vee h) + \mu(f \wedge h) = \mu(f) + \mu(h)$$

for all $f, h \in X$ such that $f \vee h, f \wedge h \in X$.

$f \vee h$: pointwise maximum, $f \wedge h$: pointwise minimum

Examples

- $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$: convex functions $f : \mathbb{R}^n \to \mathbb{R}$ (loc. uniform conv.)
- $\mathcal{M}(\mathbb{R}^n) := (C_c(\mathbb{R}^n))'$: signed Radon measures on $\mathbb{R}^n$ (weak-$*$ conv.)

Example (real Monge-Ampère operator)

$\mathrm{MA} : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n) \to \mathcal{M}(\mathbb{R}^n)$

$$f \mapsto \left[B \mapsto \int_B \det(D^2 f(x)) dx, \quad B \text{ Borel set} \right]$$

extends uniquely by continuity to a continuous valuation.

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mixed Monge-Ampère operators:

$$\mathrm{MA}(f_1, \ldots, f_n) := \frac{1}{n!} \frac{\partial^n}{\partial \lambda_1 \ldots \partial \lambda_n} \Big|_0 \mathrm{MA} \left(\sum_{i=1}^n \lambda_i f_i \right)$$

Further examples

Theorem (Alesker 2019)

For every $B \in C_c(\mathbb{R}^n)$, $A_1, \ldots, A_{n-k} \in C_c(\mathbb{R}^n, \mathrm{Sym}^2 \mathbb{R}^n)$ there exists a unique continuous valuation $\mu : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ s.t.

$$\mu(f) = \int_{\mathbb{R}^n} B(x) \det(D^2 f(x)[k], A_1(x), \ldots, A_{n-k}(x)) dx$$

for all $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n)$. det*: mixed discriminant*

Further examples

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Remark:

- μ is dually epi-translation invariant:

$$\mu(f + \ell) = \mu(f) \quad \text{for } f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}), \ell : \mathbb{R}^n \to \mathbb{R} \text{ affine}$$

- μ is k-homogeneous: $\mu(tf) = t^k \mu(f)$, $t \geq 0$

Dually epi-translation invariant valuations

F: locally convex vector space

Definition

$\mathrm{VConv}(\mathbb{R}^n, F)$: continuous, dually epi-translation invariant valuations
$$\mu : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to F.$$

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Theorem (Colesanti-Ludwig-Mussnig 2020, K. 2021)

$$\mathrm{VConv}(\mathbb{R}^n, F) = \bigoplus_{k=0}^{n} \mathrm{VConv}_k(\mathbb{R}^n, F)$$

$\mathrm{VConv}_k(\mathbb{R}^n, F)$: k-homogeneous valuations, $\mu(tf) = t^k \mu(f)$

Valuations of degree *n*

Theorem (Colesanti-Ludwig-Mussnig 2020)

$\mu \in \mathrm{VConv}_n(\mathbb{R}^n, \mathbb{R})$ iff there exists $\zeta \in C_c(\mathbb{R}^n)$ s.t.

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Theorem (K. 2023+)

Let $\Psi \in \mathrm{VConv}_n(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ *be* **locally determined**, *that is, for* $U \subset \mathbb{R}^n$ *open,* $f, h \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$:

$$f|_U = h|_U \quad \Rightarrow \quad \Psi(f)|_U = \Psi(h)|_U.$$

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Then there exists $\psi \in C(\mathbb{R}^n)$ s.t. for $B \subset \mathbb{R}^n$ bounded Borel set

$$\Psi(f)[B] = \int_B \psi(x) d\mathrm{MA}(f)[x].$$

Characterization of the real Monge-Ampère operator

$\Psi \in \mathrm{VConv}(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ is **translation equivariant** iff for $x \in \mathbb{R}^n$, $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$:

$$\Psi(f(\cdot + x))[B] = \Psi(f)[B + x] \quad \text{for all bounded Borel sets } B \subset \mathbb{R}^n.$$

Corollary (K. 2023+)

$\Psi \in \mathrm{VConv}_n(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n))$ *is locally determined and translation equivariant iff* $\Psi = c \cdot \mathrm{MA}$ *for some* $c \in \mathbb{R}$.

Translation equivariant Monge-Ampère operators

Definition

$$\mathrm{MAVal}(\mathbb{R}^n) := \{\Psi \in \mathrm{VConv}(\mathbb{R}^n, \mathcal{M}(\mathbb{R}^n)) : \Psi \text{ locally determined and}$$
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Corollary

$$\mathrm{MAVal}(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \mathrm{MAVal}_k(\mathbb{R}^n)$$

- $\mathrm{MAVal}_0(\mathbb{R}^n)$ is spanned by the Lebesgue measure.
- $\mathrm{MAVal}_n(\mathbb{R}^n)$ is spanned by MA.

Construction of valuations with the differential cycle

- For $f \in C^2(\mathbb{R}^n)$

$$\mathrm{graph}(df) := \{(x, df(x)) \in \mathbb{R}^n \times (\mathbb{R}^n)^* : x \in \mathbb{R}^n\} \subset T^*\mathbb{R}^n$$

  is an oriented C^1-submanifold of the cotangent bundle
  $\pi : T^*\mathbb{R}^n \to \mathbb{R}^n$.

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Theorem (K. 2023+)

For any differential form $\tau \in \Lambda^{n-k,k} := \Lambda^{n-k}\mathbb{R}^n \otimes \Lambda^k(\mathbb{R}^n)^*$,

$$\Psi_\tau(f)[B] := D(f)\left[1_{\pi^{-1}(B)}\tau\right] = \int_{\mathrm{graph}(df) \cap \pi^{-1}(B)} \tau$$

defines an element of $\mathrm{MAVal}_k(\mathbb{R}^n)$.

Construction of valuations with the differential cycle

Example

For standard coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$

$$\int_B \det(D^2 f(x)) dx = \int_{\mathrm{graph}(df) \cap \pi^{-1}(B)} dy_1 \wedge \cdots \wedge dy_n$$

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$$\int_B \det(D^2 f(x)) dx = \int_{\mathrm{graph}(df) \cap \pi^{-1}(B)} dy_1 \wedge \cdots \wedge dy_n$$

and

$$\int_B 1 dx = \int_{\mathrm{graph}(df) \cap \pi^{-1}(B)} dx_1 \wedge \cdots \wedge dx_n$$

for $f \in C^2(\mathbb{R}^n)$.

Translation equivariant Monge-Ampère operators

Theorem (K. 2023+)

For a continuous map $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^n)$ the following are equivalent:

1. $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$.

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3. *There exists a linear combination P of $(k \times k)$-minors such that*

$$\Psi(f)[B] = \int_B P(D^2 f(x))dx \quad \text{for } f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n).$$

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4. Ψ *is a linear combination of the mixed Monge-Ampère operators*

$$f \mapsto \mathrm{MA}(f[k], A_1, \ldots, A_{n-k}), \quad A_j \text{ quadratic polynomial}.$$

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In particular, $\dim \mathrm{MAVal}_k(\mathbb{R}^n) = \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{n-k-1}$.

Restrictions to subspaces

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Lemma

Let $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$. For $E \in \mathrm{Gr}_k(\mathbb{R}^n)$ there exists $\mathrm{Kl}_\Psi(E) \in \mathbb{R}$ s.t.

$$\Psi(\pi_E^* f) = \mathrm{Kl}_\Psi(E) \cdot \mathrm{MA}_E(f) \otimes \mathrm{vol}_{E^\perp} \quad \text{for all } f \in \mathrm{Conv}(E, \mathbb{R}).$$

$\pi_E : \mathbb{R}^n \to E$ orthogonal projection, MA_E : Monge-Ampère operator on E.

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Sketch of proof.

For Borel sets $A \subset E$, $B \subset E^\perp$,

$$(A, B) \mapsto \Psi(\pi_E^* f)[A \times B]$$

is a translation invariant measure in B

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Sketch of proof.

For Borel sets $A \subset E$, $B \subset E^\perp$,

$$(A, B) \mapsto \Psi(\pi_E^* f)[A \times B]$$

is a translation invariant measure in B, so for $f \in \mathrm{Conv}(E, \mathbb{R})$

$$(f, A) \mapsto \Psi(\pi_E^* f)[A \times \cdot] = \tilde{\Psi}(f)[A] \cdot \mathrm{vol}_{E^\perp},$$

where $\tilde{\Psi} \in \mathrm{MAVal}_k(E) = \mathbb{R} \cdot \mathrm{MA}_E$. $\qquad \square$

The Goodey-Weil distributions

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$: $\bar{\mu}(f_1, \ldots, f_k) := \frac{1}{k!} \frac{\partial^k}{\partial \lambda_1 \ldots \partial \lambda_k}\big|_0 \mu\left(\sum_{i=1}^k \lambda_i f_i\right)$

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Theorem (K. 2021)

*For every $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$ there exists a unique symmetric distribution $\mathrm{GW}(\mu) \in \mathcal{D}((\mathbb{R}^n)^k)$ **with compact support** such that*

$$\mathrm{GW}(\mu)[f_1 \otimes \cdots \otimes f_k] = \bar{\mu}(f_1, \ldots, f_k)$$

for all $f_1, \ldots, f_k \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n)$.

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- $\mathrm{GW}(\mu)[f \otimes \cdots \otimes f] = \mu(f)$ for $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n)$
- original construction due to Goodey and Weil for $\mathrm{Val}_k(\mathbb{R}^n)$

The Fourier transform of Goodey-Weil distributions

Recall: $\mathrm{GW}(\mu)$ compactly supported distribution on $(\mathbb{R}^n)^k$,
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- The Fourier transform $\mathcal{F}(\mathrm{GW}(\mu))$ is an **entire function** on $(\mathbb{C}^n)^k$:

$$\mathcal{F}(\mathrm{GW}(\mu))[z_1, \ldots, z_k] = \mathrm{GW}(\mu)[\exp(i\langle z_1, \cdot \rangle) \otimes \cdots \otimes \exp(i\langle z_k, \cdot \rangle)].$$

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- It is determined by the restriction to $(i\mathbb{R}^n)^k$. For $y_1, \ldots, y_k \in \mathbb{R}^n$:

$$\mathcal{F}(\mathrm{GW}(\mu))[iy_1, \ldots, iy_k] = \frac{1}{k!} \frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\Big|_0 \mu\left(\sum_{j=1}^k \lambda_j \exp(-\langle y_j, \cdot \rangle)\right).$$

- Right hand side: evaluation in functions defined on $\mathrm{span}(y_1, \ldots, y_k)$.

Restriction to subspaces

1. Take the orthogonal projection $\pi_E : \mathbb{R}^n \to E$.
2. Consider the **restriction** $\pi_{E*}\mu \in \mathrm{VConv}(E, \mathbb{R})$,

$$[\pi_{E*}\mu](f) := \mu(\pi_E^* f), \quad f \in \mathrm{Conv}(E, \mathbb{R}),$$

of $\mu \in \mathrm{VConv}(\mathbb{R}^n, \mathbb{R})$ to $E \in \mathrm{Gr}_k(\mathbb{R}^n)$.

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Corollary

For $\mu \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$ the following are equivalent:

1. $\mu = 0$,
2. $\pi_{E*}\mu = 0$ for all $E \in \mathrm{Gr}_k(\mathbb{R}^n)$.

In other words, μ is uniquely determined by its restrictions.

Application to measure-valued valuations

For $\phi \in C_c(\mathbb{R}^n)$, $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$, consider $\Psi[\phi] \in \mathrm{VConv}_k(\mathbb{R}^n, \mathbb{R})$:

$$\Psi[\phi](f) := \int_{\mathbb{R}^n} \phi \, d\Psi(f).$$

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For $y_1, \ldots, y_k \in E \in \mathrm{Gr}_k(\mathbb{R}^n)$:

$$\mathcal{F}(\mathrm{GW}(\Psi[\phi]))[iy_1, \ldots, iy_k] = \mathrm{Kl}_\Psi(E) \frac{\det_k \left(\langle y_j, y_l \rangle \right)_{j,l=1}^k}{(-1)^k k!} \mathcal{F}(\phi) \left(\sum_{j=1}^k iy_j \right).$$

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$= \mathcal{F}(\mathrm{GW}(\mathrm{Kl}_\Psi(E)\mathrm{MA}_E \otimes \mathrm{vol}_{E^\perp}[\phi]))[iy_1, \ldots, iy_k]$ $\qquad \square$

From valuations to polynomials

Theorem (K. 2023+)

For every $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$ there exists a unique polynomial $Q[\Psi]$ on $(\mathbb{C}^n)^k$ of degree at most $2k$ such that for all $\phi \in C_c(\mathbb{R}^n)$

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$Q[\Psi](z_1, \ldots, z_k) = P(\sum_{i=1}^{k} z_i \cdot z_i^T)$, *where P is a linear combination of $(k \times k)$-minors.*

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$(k \times k)$-minors: irreducible representation of $\mathrm{GL}(n, \mathbb{R})$.

Irreducibility under $\mathrm{GL}(n, \mathbb{R})$

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The map $\mathrm{MAVal}_k(\mathbb{R}^n) \ni \Psi \mapsto Q[\Psi]$ defines a $\mathrm{GL}(n, \mathbb{R})$-equivariant map into an irreducible representation of $\mathrm{GL}(n, \mathbb{R})$.

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Lemma

The following valuations span $\mathrm{GL}(n, \mathbb{R})$-invariant subspaces of $\mathrm{MAVal}_k(\mathbb{R}^n)$:

1. $f \mapsto D(f)[1_{\pi^{-1}(\cdot)}\tau]$, $\tau \in \Lambda^{n-k,k}$.
2. $f \mapsto \mathrm{MA}(f[k], A_1, \ldots, A_{n-k})$, A_j *quadratic polynomial.*

Thus, these spaces coincide with $\mathrm{MAVal}_k(\mathbb{R}^n)$.

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Thus, these spaces coincide with $\mathrm{MAVal}_k(\mathbb{R}^n)$.

Left to do: $(f, B) \mapsto \int_B P(D^2 f(x))dx$, P combination of $(k \times k)$-minors

From $(k \times k)$-minors to differential forms

If $\tau \in \Lambda^{n-k,k}$, then

$$D(f)[1_{\pi^{-1}(B)}\tau] = \int_B P_\tau(D^2 f(x)) dx$$

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Corollary

If P is a linear combination of $(k \times k)$-minors, then there exists $\tau \in \Lambda^{n-k,k}$ s.t.

$$D(f)[1_{\pi^{-1}(B)}\tau] = \int_B P(D^2 f(x))dx.$$

Translation equivariant Monge-Ampère operators

Theorem (K. 2023+)

For a continuous map $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^n)$ the following are equivalent:

1. $\Psi \in \mathrm{MAVal}_k(\mathbb{R}^n)$.

2. $\Psi(f)[B] = D(f)[1_{\pi^{-1}(B)}\tau]$ *for some* $\tau \in \Lambda^{n-k,k}$.

3. *There exists a linear combination P of $(k \times k)$-minors such that*

$$\Psi(f)[B] = \int_B P(D^2 f(x))dx \quad \text{for } f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n).$$

4. Ψ *is a linear combination of the mixed Monge-Ampère operators*

$$f \mapsto \mathrm{MA}(f[k], A_1, \ldots, A_{n-k}), \quad A_j \text{ quadratic polynomial.}$$

Thank you for your attention!