Equivariant Valuations of Convex Functions

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Affine constructions on convex bodies

- Difference body $D : \mathcal{K}^n \to \mathcal{K}^n$

$$DK = K + (-K) = \{x - y : \, x, y \in K\}$$

$\mathcal{K}^n$ ... convex bodies (compact, convex subsets of $\mathbb{R}^n$)

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- Projection body $\Pi : \mathcal{K}^n \to \mathcal{K}^n$

$$V_1(\Pi K | \operatorname{span}\{u\}) = 2V_{n-1}(K|u^\perp), \quad u \in \mathbb{S}^{n-1}$$

$\mathcal{K}^n$	...	convex bodies (compact, convex subsets of $\mathbb{R}^n$)
V_i	...	volume (on $\mathbb{R}^i$)
$K\|E$	...	orthogonal projection onto subspace E

Affine constructions on convex bodies

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D and Π are Minkowski valuations:

Definition

A continuous map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Minkowski valuation, if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L), \quad \forall K, L, K \cup L \in \mathcal{K}^n$$

continuity ... w.r.t. Hausdorff metric

Characterization of D and Π

Theorem (Ludwig 2005)

Suppose that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a translation-invariant Minkowski valuation.

- Φ *is* $\mathrm{SL}_n(\mathbb{R})$*-equivariant* $\quad\Leftrightarrow\quad$ $\Phi = cD$, $c \geq 0$.

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Our setting

Finite convex functions

$$\mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) := \{f : \mathbb{R}^n \to \mathbb{R} : \ f \text{ convex}\}$$

with

- epi-convergence (i.e. uniform convergence on compact sets)
- $\mathrm{SL}_n(\mathbb{R})$-action: $\quad (\varphi \cdot f)(x) = f(\varphi^{-1}x), \quad \varphi \in \mathrm{SL}_n(\mathbb{R})$

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Example

Support functions of $K \in \mathcal{K}^n$

$$h_K(x) = \sup_{y \in K} \langle x, y \rangle, \quad x \in \mathbb{R}^n$$

Valuations on convex functions

We want to consider maps $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, which are

- continuous
- dually epi-translation-invariant: $\forall \lambda$ affine

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- $\mathrm{SL}_n(\mathbb{R})$-equi-/contravariant

Previous results

Ludwig 2012:	$\Psi : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$
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Theorem (H. & Knoerr 2023)

Suppose that $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ is continuous and additive.

Ψ *is* $\mathrm{SL}_n(\mathbb{R})$*-equivariant*

$\Leftrightarrow \quad \exists c \in \mathbb{R}, \nu \in \mathcal{M}_c^+(\mathbb{R})$ *with* $\int_{\mathbb{R}^\times} |s|^{-1} d\nu(s) < \infty$, *s.t.*

$$\Psi(f)[x] = c \cdot f(0) + \int_{\mathbb{R}^\times} \frac{f(sx) - f(0)}{s^2} d\nu(s), \quad x \in \mathbb{R}^n,$$

for every $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$.

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e.g. $\Psi(f)[x] = f(x) + f(-x)$, $x \in \mathbb{R}^n$

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Equivalant valuations on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$

Theorem (H. & Knoerr 2023+)

Suppose that $\Psi : \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \to \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ *is a continuous, dually epi-translation-invariant valuation.*

Ψ *is* $\mathrm{SL}_n(\mathbb{R})$*-equivariant* $\quad\Leftrightarrow\quad$ $\exists c \in \mathbb{R}, \nu \in \mathcal{M}_c^+(\mathbb{R})$, *s.t.*

$$\int_{\mathbb{R}^\times} |s|^{-1} d\nu(s) < \infty \quad \text{and} \quad \int_{\mathbb{R}^\times} s^{-1} d\nu(s) = 0$$

and

$$\Psi(f)[x] = c + \int_{\mathbb{R}^\times} \frac{f(sx) - f(0)}{s^2} d\nu(s), \quad x \in \mathbb{R}^n,$$

for every $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$.

Notes on the proof

- $\Psi_x : f \mapsto \Psi(f)[x]$ is a valuation on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$, $x \neq 0 \in \mathbb{R}^n$

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- Goodey–Weil distribution for $\mathrm{VConv}_k(\mathbb{R}^n)$ (Knoerr 2021)

$$\mathrm{GW} : \mathrm{VConv}_k(\mathbb{R}^n) \to \mathcal{D}'_c((\mathbb{R}^n)^k)$$

$\mathrm{VConv}_k(\mathbb{R}^n)$	...	epi-continuous, k-homogeneous, dually epi-translation-invariant valuations
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- compact support $\implies$ support is 1-dimensional

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Theorem (H. & Knoerr 2023+)

Suppose that $\mu \in \mathrm{VConv}_k(\mathbb{R}^n)$ *and* $E \in \mathrm{Gr}_i(\mathbb{R}^n)$, $0 \leq i \leq n-1$.

If $\operatorname{supp} \mathrm{GW}(\mu) \subseteq \Delta(E) \implies \exists \mu_E \in \mathrm{VConv}_k(E)$, *s.t.*

$$\mu(f) = \mu_E(f|_E),$$

for all $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$.

$\Delta(y) = (y, \dots, y) \in (\mathbb{R}^n)^k$

Idea: $\mathrm{GW}(\mu)[\varphi]$ cannot depend on "normal" derivatives

Notes on the proof

$$\implies \Psi_x = \Psi_x^0 + \Psi_x^1, \text{ with } \Psi_x^i \in \mathrm{VConv}_i(\mathbb{R}^n)$$

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$\square$

Contravariant valuations on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$

Suppose now that Ψ is $\mathrm{SL}_n(\mathbb{R})$-contravariant.

- $\operatorname{supp} \mathrm{GW}(\Psi_x)$ is compact!
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Thank you for your attention!