The Lp Higher-order Petty projection inequality

J. Haddad, D. Langharst, D. Ye, E. Putterman, M. Roysdon

Convex Geometry - Analytic Aspects

Definition ($K \subseteq \mathbb{R}^n, \Pi K \subseteq \mathbb{R}^n$)

$$h_{\Pi K}(x) = \int_{S^{n-1}} \frac{1}{2} |\langle x, v \rangle| dS_K(v)$$

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$$h_{\Pi_p K}(x)^p = \int_{S^{n-1}} |\langle x, v \rangle|^p dS_{p,K}(v)$$

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Definition ($K \subseteq \mathbb{C}^n, \Pi_C K \subseteq \mathbb{C}^n, C \subseteq \mathbb{C}$, Haberl (2019))

$$h_{\Pi_C K}(x) = \int_{S^{2n-1}} h_C(v \cdot x) dS_K(v)$$

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Definition ($K \subseteq \mathbb{R}^n, \Pi_m K \subseteq (\mathbb{R}^n)^m$, H.L.Y.P.R (2023))

$$h_{\Pi_m K}(x) = \int_{S^{n-1}} \max_{i=1\ldots m} \langle x_i, v\rangle_{-} dS_K(v)$$

Definition $(K \subseteq \mathbb{R}^n, \Pi_{Q,p}K \subseteq \mathrm{M}_{n,m}(\mathbb{R}), Q \subseteq \mathbb{R}^m)$

$$h_{\Pi_{Q,p}K}(x)^p = \int_{S^{n-1}} h_Q(v^t.x)^p dS_{p,K}(v)$$

Theorem

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$$\begin{pmatrix} & T & \end{pmatrix} \cdot \left(\begin{array}{c|c|c|c} v_1 & v_2 & \cdots & v_m \end{array}\right) = \left(\begin{array}{c|c|c|c} Tv_1 & Tv_2 & \cdots & Tv_m \end{array}\right)$$

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$$\left(v_1 \,\middle|\, v_2 \right) \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \left(av_1 + bv_2 \right)$$

$$\left(v \right) \cdot \begin{pmatrix} a & b \end{pmatrix} = \left(av \,\middle|\, bv \right)$$

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6. $\Pi_{Q.B,p}K = (\Pi_{Q,p}K).B$
7. For $p \to \infty$ we get operator norms.

Operator norms

Theorem

Let $B_{E,F} \subseteq \mathrm{M}_{n,m}(\mathbb{R})$ be the unit ball in operator norm, of the maps between banach spaces

$$(\mathbb{R}^n, E) \to (\mathbb{R}^m, F)$$

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then

$$|B_{E,F}|_{nm} \leq |B_{B_E,F}|_{nm} = |E|_n^{-m} \, |B_2^n|_n^m \, \left|B_{B_2^n,F}\right|_{nm}$$

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$$|B_{E,F}|_{nm} \leq |E|_n^{-m} |F|_n^n |B|_n^{m-n} |S_\infty|_{nm}$$

Fiber Symmetrization

Let $v \in S^{n-1}$.

$$\langle v\rangle^m = \langle v\rangle \times \cdots \times \langle v\rangle = (\lambda_1 v, \ldots, \lambda_m v) = v.\,\mathrm{M}_{m,1}(\mathbb{R})$$

$$\langle v\rangle^{\perp m} = \langle v\rangle^\perp \times \cdots \times \langle v\rangle^\perp = \{x \in \mathrm{M}_{n,m}(\mathbb{R}) : v^t x = 0\}$$

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Theorem

$$\bar{S}_v \Pi^{\circ}_{Q,p} K \subseteq \Pi^{\circ}_{Q,p} S_v K$$

Thank you