Threshold for the expected measure of random polytopes

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Convex Geometry – Analytic Aspects

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Threshold for the volume

The original, still vague, formulation:

- Let μ be a Borel probability measure on $\mathbb{R}^n$ such that $K = \mathrm{conv}(\mathrm{supp}(\mu))$ is a convex body in $\mathbb{R}^n$.
- Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent random vectors X_i distributed according to μ.

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- For each $N > n$ consider the random polytope

$$K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$$

  and the normalized expectation of its volume

$$E_\mu(N) = \mathbb{E}_{\mu^N}\left(\frac{|K_N|}{|K|}\right).$$

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$$E_\mu(N) = \mathbb{E}_{\mu^N}\left(\frac{|K_N|}{|K|}\right).$$

- We say that μ exhibits a threshold at $\varrho > 0$ if

$$E_\mu(N) \sim 0 \quad \text{when} \quad N \ll \exp(\varrho n)$$

and

$$E_\mu(N) \sim 1 \quad \text{when} \quad N \gg \exp(\varrho n).$$

Threshold for the volume

Dyer-Füredi-McDiarmid (1992)

Let μ_n be the uniform measure on $\{-1,1\}^n$ (then, $K = \mathrm{conv}(\mathrm{supp}(\mu_n)) = [-1,1]^n$). If $\varrho = \ln 2 - \frac{1}{2}$ (the same for every n) then for every $\epsilon \in (0,\varrho)$ we have that

$$\lim_{n\to\infty} \sup\left\{2^{-n}\mathbb{E}\,|K_N| \colon N \leqslant \exp((\varrho-\epsilon)n)\right\} = 0$$

and

$$\lim_{n\to\infty} \inf\left\{2^{-n}\mathbb{E}\,|K_N| \colon N \geqslant \exp((\varrho+\epsilon)n)\right\} = 1.$$

- Dyer, Füredi and McDiarmid also obtained a similar result for the case where μ_n is the uniform measure on $[-1,1]^n$. The value of the constant is now

$$\varrho = \ln(2\pi) - (\gamma + 1/2),$$

where γ is Euler's constant.

Threshold for the volume

- A very general guess is that the following might be true.

Statement

Let $\delta \in \left(0, \frac{1}{2}\right)$. There exists a sequence $\epsilon_n(\delta) \to 0$ as $n \to \infty$ and $n_0(\delta) \in \mathbb{N}$ such that if $n \geqslant n_0$ and K is a convex body in $\mathbb{R}^n$ then we may find a constant $\varrho = \varrho(K)$ such that

$$\sup \left\{ \mathbb{E}\left(\frac{|K_N|}{|K|}\right) : N \leqslant \exp((1-\epsilon_n(\delta))\varrho n) \right\} \leqslant \delta$$

and

$$\inf \left\{ \mathbb{E}\left(\frac{|K_N|}{|K|}\right) : N \geqslant \exp((1+\epsilon_n(\delta))\varrho n) \right\} \geqslant 1-\delta,$$

where $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$ and X_i are independent random points uniformly distributed in K.

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where $K_N = \mathrm{conv}\{X_1, \ldots, X_N\}$ and X_i are independent random points uniformly distributed in K.

- The first main question is to identify the constant $\varrho(K)$ and then to establish these estimates with a constant $\epsilon_n(\delta)$, ideally independent from K, which tends to 0 as $n \to \infty$.
- The "window" of the threshold is the quantity $2\epsilon_n(\delta)\varrho$.

Related works

Apart from the results of Dyer, Füredi and McDiarmid, sharp results in the spirit of the statement above are known only in (very) special cases.

- M. E. Dyer, Z. Füredi and C. McDiarmid, *Volumes spanned by random points in the hypercube*, Random Structures Algorithms 3 (1992), 91–106.
- P. Pivovarov, *Volume thresholds for Gaussian and spherical random polytopes and their duals*, Studia Math. 183 (2007), no. 1, 15–34.
- D. Gatzouras and A. Giannopoulos, *Threshold for the volume spanned by random points with independent coordinates*, Israel J. Math. 169 (2009), 125–153.
- G. Bonnet, G. Chasapis, J. Grote, D. Temesvari and N. Turchi, *Threshold phenomena for high-dimensional random polytopes*, Commun. Contemp. Math. 21 (2019), no. 5, 1850038, 30 pp.
- G. Bonnet, Z. Kabluchko and N. Turchi, *Phase transition for the volume of high-dimensional random polytopes*, Random Structures Algorithms 58 (2021), no. 4, 648–663.
- A. Frieze, W. Pegden and T. Tkocz, *Random volumes in d-dimensional polytopes*, Discrete Anal. 2020, Paper No. 15, 17 pp.
- D. Chakraborti, T. Tkocz and B-H. Vritsiou, *A note on volume thresholds for random polytopes*, Geom. Dedicata 213 (2021), 423–431.

A more general problem

- Let μ be a log-concave probability measure μ on $\mathbb{R}^n$ with barycenter at the origin.
- It is known that if a probability measure μ is log-concave and $\mu(H) < 1$ for every hyperplane H in $\mathbb{R}^n$, then μ has a log-concave density f_μ.
- By the Brunn-Minkowski inequality, the uniform measure on a convex body K in $\mathbb{R}^n$ is log-concave.
- Let $X_1, X_2, \dots$ be independent random points in $\mathbb{R}^n$ distributed according to μ and for any $N > n$ define the random polytope

$$K_N = \mathrm{conv}\{X_1, \dots, X_N\}.$$

- Consider the expectation $\mathbb{E}_{\mu^N}[\mu(K_N)]$ of the measure of K_N, where $\mu^N = \mu \times \cdots \times \mu$, N times.
- If $\mu = \mu_K$, the uniform measure on a convex body K in $\mathbb{R}^n$, then this quantity is the same with the one we discussed before.

A more general problem

- Given $\delta \in \left(0, \frac{1}{2}\right)$ we say that μ satisfies a "δ-upper threshold" with constant ϱ_1 if

$$\sup\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \leqslant \exp(\varrho_1 n)\} \leqslant \delta \tag{1}$$

and that μ satisfies a "δ-lower threshold" with constant ϱ_2 if

$$\inf\{\mathbb{E}_{\mu^N}[\mu(K_N)] : N \geqslant \exp(\varrho_2 n)\} \geqslant 1 - \delta. \tag{2}$$

- Then, we define

$$\varrho_1(\mu, \delta) = \sup\{\varrho_1 : (1) \text{ holds true}\} \quad \text{and} \quad \varrho_2(\mu, \delta) = \inf\{\varrho_2 : (2) \text{ holds true}\}.$$

- Our main goal is to obtain upper bounds for the difference

$$\pi(\mu, \delta) := \varrho_2(\mu, \delta) - \varrho_1(\mu, \delta)$$

for any fixed $\delta \in \left(0, \frac{1}{2}\right)$.

Tukey's half-space depth

- Let μ be a probability measure on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$ we denote by $\mathcal{H}(x)$ the set of all half-spaces H of $\mathbb{R}^n$ containing x.
- Tukey's half-space depth is the function

$$\varphi_\mu(x) = \inf\{\mu(H) : H \in \mathcal{H}(x)\}.$$

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G.-Brazitikos-Pafis (2022)

If μ is a log-concave probability measure on $\mathbb{R}^n$ then

$$\exp(-c_1 n) \leqslant \int_{\mathbb{R}^n} \varphi_\mu(x)\, d\mu(x) \leqslant \exp\left(-c_2 n/L_\mu^2\right),$$

where $c_1, c_2 > 0$ are absolute constants and L_μ is the isotropic constant of μ.

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where $c_1, c_2 > 0$ are absolute constants and L_μ is the isotropic constant of μ.

- The right-hand side inequality answers a question from "math-overflow" and combined with the left-hand side inequality determines the expectation of φ since now it is known that $L_\mu \leqslant c\sqrt{\ln n}$.

The geometric lemmas

- For any convex body $B \subset \mathbb{R}^n$ we define

$$\varphi_1(B) = \sup_{x \notin B} \varphi_\mu(x) \quad \text{and} \quad \varphi_2(B) = \inf_{x \in B} \varphi_\mu(x).$$

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The first lemma

Let $B \subset \mathbb{R}^n$ be a convex body. For every $N > n$ we have

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(B) + N\,\varphi_1(B).$$

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$$\mathbb{E}_{\mu^N}(\mu(K_N)) \leqslant \mu(B) + N\,\varphi_1(B).$$

The second lemma

Let $B \subset \mathbb{R}^n$ be a convex body. For every $N > n$ we have

$$\mathbb{E}_{\mu^N}(\mu(K_N)) \geqslant \mu(B)\left(1 - 2\binom{N}{n}(1 - \varphi_2(B))^{N-n}\right).$$

The geometric lemmas

- The idea is to define an increasing family of convex bodies $\{B_t\}_{t>0}$, depending on μ, and to identify a value ϱ so that, for some small $\epsilon = \epsilon(\mu, \delta)$, the following hold:
  1. $\mu(B_{(1-\epsilon)\varrho n}) \leqslant \delta/2$
  2. $\mu(B_{(1+\epsilon)\varrho n}) \geqslant 1 - \delta/2$
  3. $\exp((1-2\epsilon)\varrho n)\, \varphi_1(B_{(1-\epsilon)\varrho n}) \leqslant \delta/2$
  4. $N \geqslant \exp((1+2\epsilon)\varrho n) \implies 2\binom{N}{n}(1-\varphi_2(B_{(1+\epsilon)\varrho n}))^{N-n} \leqslant \delta/2.$

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    4. $N \geqslant \exp((1+2\epsilon)\varrho n) \implies 2\binom{N}{n}(1-\varphi_2(B_{(1+\epsilon)\varrho n}))^{N-n} \leqslant \delta/2.$
- Then, the first lemma shows that $\varrho_1(\mu,\delta) \geqslant (1-2\epsilon)\varrho n$ and the second lemma shows that $\varrho_2(\mu,\delta) \leqslant (1+2\epsilon)\varrho n$.
- So, we have a threshold at ϱ with $\pi(\mu,\delta) := \varrho_2(\mu,\delta) - \varrho_1(\mu,\delta) \leqslant 4\epsilon\varrho$.

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- Then, the first lemma shows that $\varrho_1(\mu,\delta) \geqslant (1-2\epsilon)\varrho n$ and the second lemma shows that $\varrho_2(\mu,\delta) \leqslant (1+2\epsilon)\varrho n$.
- So, we have a threshold at ϱ with $\pi(\mu,\delta) := \varrho_2(\mu,\delta) - \varrho_1(\mu,\delta) \leqslant 4\epsilon\varrho$.
- If all this is going to work, we must have that

$$\varphi_1(B_t) \text{ is close to } \varphi_2(B_t).$$

The family $\{B_t\}_{t>0}$

- Let μ be a centered log-concave probability measure on $\mathbb{R}^n$ with density $f := f_\mu$.
- The logarithmic Laplace transform of μ on $\mathbb{R}^n$ is defined by

$$\Lambda_\mu(\xi) = \ln\Big(\int_{\mathbb{R}^n} e^{\langle \xi, z\rangle} f(z)dz\Big).$$

- It is easily checked that Λ_μ is convex and $\Lambda_\mu(0) = 0$: Since $\mathrm{bar}(\mu) = 0$, Jensen's inequality shows that $\Lambda_\mu(\xi) \geqslant 0$ for all ξ.
- We define

$$\Lambda_\mu^*(x) = \sup_{\xi\in\mathbb{R}^n}\left\{\langle x, \xi\rangle - \Lambda_\mu(\xi)\right\}.$$

  In other words, Λ_μ^* is the Legendre transform of Λ_μ.

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The family $\{B_t\}_{t>0}$

For every $t > 0$ we also set
$$B_t(\mu) := \{x \in \mathbb{R}^n : \Lambda_\mu^*(x) \leqslant t\}.$$

Estimates for $\varphi_1(B_t(\mu))$ and $\varphi_2(B_t(\mu))$

Upper bound

Let μ be a Borel probability measure on $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ we have

$$\varphi_\mu(x) \leqslant \exp(-\Lambda_\mu^*(x)).$$

In particular, for any $t > 0$ we have that $\varphi_1(B_t(\mu)) \leqslant \exp(-t)$.

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Lower bound: G.-Brazitikos-Pafis (2022)

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $t > 0$ we have that

$$\varphi_2(B_t(\mu_K)) = \inf\{\varphi_{\mu_K}(x) : x \in B_t(\mu_K)\} \geqslant \frac{1}{10}\exp(-t - 2\sqrt{n}),$$

where μ_K is Lebesgue measure on K.

Comparison of φ_{μ_K} and $\exp(-\Lambda^*_{\mu_K})$

- Setting $\omega_{\mu_K}(x) = \ln\left(\frac{1}{\varphi_{\mu_K}(x)}\right)$, we have actually proved a two-sided inequality.

Uniform measure on a convex body

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Then, for every $x \in \mathrm{int}(K)$ we have that

$$\omega_{\mu_K}(x) - 5\sqrt{n} \leqslant \Lambda^*_{\mu_K}(x) \leqslant \omega_{\mu_K}(x). \tag{3}$$

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First question

Is it true that an analogue of (3) holds true for any centered log-concave probability measure μ on $\mathbb{R}^n$?

The value of ϱ

- As we will see, for any centered log-concave probability measure μ on $\mathbb{R}^n$, the correct value of ϱ is

$$\varrho = \frac{1}{n}\mathbb{E}_\mu(\Lambda_\mu^*).$$

The value of ϱ

- As we will see, for any centered log-concave probability measure μ on $\mathbb{R}^n$, the correct value of ϱ is
$$\varrho = \frac{1}{n}\mathbb{E}_\mu(\Lambda_\mu^*).$$
- Consider the parameter
$$\beta(\mu) = \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{(\mathbb{E}_\mu(\Lambda_\mu^*))^2}.$$
- Roughly speaking, the plan is the following: provided that φ_μ is "almost constant" on $\partial(B_t(\mu))$ for all $t > 0$ and that $\beta(\mu) = o_n(1)$, we can establish a "sharp threshold" at ϱ for the expected measure of K_N with window
$$\pi(\mu,\delta) \leqslant c\sqrt{\beta(\mu)/\delta}\,\varrho.$$

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$$\pi(\mu, \delta) \leqslant c\sqrt{\beta(\mu)/\delta}\,\varrho.$$
- But, first of all, we would like to know for which centered log-concave probability measures μ on $\mathbb{R}^n$ the parameter $\beta(\mu)$ is well-defined. This is true if
$$\|\Lambda_\mu^*\|_{L^2(\mu)} = \left(\mathbb{E}_\mu\left((\Lambda_\mu^*)^2\right)\right)^{1/2} < \infty.$$
- A stronger sufficient condition is to require that Λ_μ^* has finite moments of all orders.

Moments of the Cramer transform

- Given $\kappa \in (0, 1/n]$ we say that a measure μ on $\mathbb{R}^n$ is κ-concave if $\mu((1-\lambda)A + \lambda B) \geqslant ((1-\lambda)\mu^{\kappa}(A) + \lambda\mu^{\kappa}(B))^{1/\kappa}$ for all compact subsets A, B of $\mathbb{R}^n$ with $\mu(A)\mu(B) > 0$ and all $\lambda \in (0,1)$.

G.-Brazitikos-Pafis (2022)

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$. Let $\kappa \in (0, 1/n]$ and let μ be a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$. Then,

$$\int_{\mathbb{R}^n} e^{\frac{\kappa\Lambda_{\mu}^*(x)}{2}} d\mu(x) < \infty.$$

- Since μ_K is $1/n$-concave, we get that $\Lambda_{\mu_K}^*$ has finite moments of all orders.

Moments of the Cramer transform

- Besides this, we can show that Λ_μ^* has finite moments of all orders in the following cases:
  (i) If μ is a centered probability measure on $\mathbb{R}$ or a product of such measures.
  (ii) If μ is supported on a convex body and has a continuous positive density.
  (iii) If μ is a centered log-concave probability measure on $\mathbb{R}^n$ and there exists a function $g : [1,\infty) \to [1,\infty)$ with $\lim_{t\to\infty} g(t)/\ln(t+1) = +\infty$ such that $Z_t^+(\mu) \supseteq g(t) Z_2^+(\mu)$ for all $t \geqslant 2$, where $\{Z_t^+(\mu)\}_{t\geqslant 1}$ is the family of one-sided L_t-centroid bodies of μ.

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Second question

Let μ be a centered log-concave probability measure μ on $\mathbb{R}^n$. Is it true that Λ_μ^* has finite moments?

Moments of the Cramer transform: Estimates

- The method of proof of the Theorem gives in fact reasonable upper bounds for $\|\Lambda_\mu^*\|_{L^p(\mu)}$. If K is a centered convex body in $\mathbb{R}^n$ and μ is a centered κ-concave probability measure with $\mathrm{supp}(\mu) = K$ then

$$\mathbb{E}_\mu(\Lambda_\mu^*) \leqslant \left(\mathbb{E}_\mu[(\Lambda_\mu^*)^2]\right)^{1/2} \leqslant \frac{c \ln n}{\kappa} \|f_\mu\|_\infty^{1/2}.$$

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- In particular, if we assume that $\mu = \mu_K$ is the uniform measure on a centered convex body then we obtain a sharp two sided estimate for the most interesting case where $p = 1$ or 2.

Theorem

Let K be a centered convex body of volume 1 in $\mathbb{R}^n$, $n \geqslant 2$. Then,

$$c_1 n / L_{\mu_K}^2 \leqslant \|\Lambda_{\mu_K}^*\|_{L^1(\mu_K)} \leqslant \|\Lambda_{\mu_K}^*\|_{L^2(\mu_K)} \leqslant c_2 n \ln n,$$

where L_{μ_K} is the isotropic constant of the uniform measure μ_K on K.

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where L_{μ_K} is the isotropic constant of the uniform measure μ_K on K.

- The left-hand side inequality follows easily from Jensen inequality and the estimate $\int_{\mathbb{R}^n} e^{-\Lambda_\mu^*(x)}\,d\mu(x) \leqslant \exp\left(-cn/L_\mu^2\right)$ that we have proved in order to estimate the expectation of Tukey's half-space depth.
- Both the lower and the upper bound are of optimal order with respect to the dimension. This can be seen e.g. from the example of the uniform measure on the cube or the Euclidean ball.

Upper threshold: the final step

- The threshold is a consequence of Chebyshev's inequality. Recall that

$$\beta(\mu)=\frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{(\mathbb{E}_\mu(\Lambda_\mu^*))^2}$$

and $B_t(\mu)=\{x\in\mathbb{R}^n:\Lambda_\mu^*(x)\leqslant t\}$.

- Let $\varrho:=\frac{1}{n}\mathbb{E}_\mu(\Lambda_\mu^*)$. Then, for all $\epsilon\in(0,1)$, from Chebyshev's inequality we have that

$$\begin{aligned}\mu(B_{(1-\epsilon)\varrho n}(\mu))=\mu(\{\Lambda_\mu^*\leqslant(1-\epsilon)\varrho n\})&\leqslant\mu(\{|\Lambda_\mu^*-\mathbb{E}_\mu(\Lambda_\mu^*)|\geqslant\epsilon\mathbb{E}_\mu(\Lambda_\mu^*)\})\\&\leqslant\frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{\epsilon^2(\mathbb{E}_\mu(\Lambda_\mu^*))^2}=\frac{\beta(\mu)}{\epsilon^2}.\end{aligned}$$

- Choosing $\epsilon=\sqrt{2\beta(\mu)/\delta}$ we have that

$$\mu(B_{(1-\epsilon)\varrho n}(\mu))\leqslant\frac{\delta}{2}$$

and this implies that

$$\varrho_1(\mu,\delta)\geqslant(1-\sqrt{8\beta(\mu)/\delta})\varrho.$$

Lower threshold: the final step

- The argument is similar. For all $\epsilon \in (0,1)$, from Chebyshev's inequality we have that

$$1 - \mu(B_{(1+\epsilon)\varrho n}(\mu)) \leqslant \mu(\{\Lambda_\mu^* \geqslant (1+\epsilon)\varrho n\}) \leqslant \mu(\{|\Lambda_\mu^* - \mathbb{E}_\mu(\Lambda_\mu^*)| \geqslant \epsilon \mathbb{E}_\mu(\Lambda_\mu^*)\})$$
$$\leqslant \frac{\mathrm{Var}_\mu(\Lambda_\mu^*)}{\epsilon^2(\mathbb{E}_\mu(\Lambda_\mu^*))^2} = \frac{\beta(\mu)}{\epsilon^2}.$$

- Choosing $\epsilon = \sqrt{2\beta(\mu)/\delta}$ we have that

$$1 - \mu(B_{(1+\epsilon)\varrho n}(\mu)) \leqslant \frac{\delta}{2}$$

  and this implies that

$$\varrho_2(\mu,\delta) \leqslant (1 + \sqrt{8\beta(\mu)/\delta})\varrho.$$

- In a few words, the final estimate is

$$\pi(\mu,\delta) \leqslant c_1\sqrt{\beta(\mu)/\delta}\,\varrho$$

  and, in the case of a convex body, $c_1/L_{\mu_K}^2 \leqslant \varrho \leqslant c_2 \ln n$.

The main questions

1. Let μ be a log-concave probability measure μ on $\mathbb{R}^n$ and define $\omega_\mu(x) = \ln\left(\frac{1}{\varphi_\mu(x)}\right)$. Is it true that
$$\|\omega_\mu - \Lambda_\mu^*\|_\infty = o(n)?$$
A stronger question is if
$$\|\omega_\mu - \Lambda_\mu^*\|_\infty = O(\sqrt{n}).$$
This is true for the uniform measure on any n-dimensional convex body.
2. Let μ be a log-concave probability measure μ on $\mathbb{R}^n$. Is it true that Λ_μ^* has finite moments of all orders? This is true for the uniform measure, and more generally for any κ-concave measure where $0 < \kappa \leqslant \frac{1}{n}$, on any n-dimensional convex body.
3. Estimate
$$\beta_n^* := \sup\{\beta(\mu_K) : K \text{ is a centered convex body of volume 1 in } \mathbb{R}^n\}$$
or, more generally,
$$\beta_n := \sup\{\beta(\mu) : \mu \text{ is a centered log-concave probability measure on } \mathbb{R}^n\}.$$
Is it true that $\lim\limits_{n\to\infty} \beta_n = 0$?