Funk geometry of polytopes and their flags

joint work with C. Vernicos and C. Walsh

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Convex Geometry - Analytic Aspects

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Funk and Hilbert geometries

Assume $K \subset \mathbb{R}^n$ is a convex body with $\mathrm{int}(K) \neq \emptyset$.

Definition

The Funk metric on $\mathrm{int}(K)$ is the non-reversible Finsler metric whose unit tangent ball $B_x K$ is K, with x at the origin. Equivalently, $\phi_K^F|_x(v) = \|v\|_{K-x}$.

It is an affine-invariant construction. The distance is $d_K^F(x, y) = \log \frac{|xz|}{|yz|}$.

Definition

The Hilbert metric is

$$d_K^H(x, y) = \frac{1}{2}(d_K^F(x, y) + d_K^F(y, x)) = \frac{1}{2} \log \frac{|xz||wy|}{|yz||wx|}.$$

Like the cross ratio, the Hilbert metric is projectively invariant.

- Both are examples of "projective metrics": straight segments are geodesics.

- **Example.** The Funk metric in the unit Euclidean ball is $d_F(x, y) = d_H(x, y) + f(y) - f(x)$ where d_H is the Beltrami-Klein hyperbolic metric (also the Hilbert metric in the ball), and $f(x) = -\frac{1}{2} \log(1 - |x|^2)$.

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Volume in Funk geometry

The outward ball in Funk metric is

$$B_K^F(q,r) = \{x : d_K^F(q,x) \le r\} = (1-e^{-r})(K-q)+q.$$

Defintion

The Holmes-Thompson volume of $A \subset \mathrm{int}(K)$ is $\mathrm{vol}_K(A) = \omega_n^{-1}\int_A |K^x|dx$, where $K^x \subset (\mathbb{R}^n)^*$ is the polar body with respect to x.

We will consider the volume of Funk balls:

$$\mathrm{vol}_K(B_K(0,r)) = \omega_n^{-1}\int_{(1-e^{-r})K} |K^x|dx.$$

Basic properties:

- *Multiplicativity*. Assume $K \subset \mathbb{R}^a, L \subset \mathbb{R}^b$. Then
$$(a+b)!\omega_{a+b}\mathrm{vol}_{K\times L}(B_{K\times L}((p,q),r)) = a!\omega_a \mathrm{vol}_K(B_K(p,r))\cdot b!\omega_b \mathrm{vol}_L(B_L(q,r)).$$
- *Duality*. Assume $0 \in \mathrm{int}(K)$. Then $\mathrm{vol}_K(B_K(0,r)) = \mathrm{vol}_{K^\circ}(B_{K^\circ}(0,r))$.

Corollary. If H_n is a centered n-dimensional Hanner polytope, and $\lambda = 1-e^{-r}$,

$$\mathrm{vol}_{H_n}(B_{H_n}(0,r)) = \omega_n^{-1}\int_{\lambda H_n} |H_n^x|dx = \frac{2^n}{n!\omega_n}\left(\log\frac{1+\lambda}{1-\lambda}\right)^n.$$

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Projective invariance

Theorem (F)

Let $g : \mathbb{RP}^n \to \mathbb{RP}^n$ be a collineation (fractional linear map), and assume $g(K) \subset \mathbb{R}^n$.
Let ϕ_K be the Funk Finsler norm on $\mathrm{int}(K)$.
Then $g^*\phi_{gK} - \phi_K \in C(TK)$ is an exact 1-form.

Corollary

Funk volume is projectively invariant: $\mathrm{vol}_K(A) = \mathrm{vol}_{gK}(gA)$.

Furthermore, the Funk metric exhibits projective duality.

For $K \subset \mathbb{RP}^n$, $K^\vee = \{\xi \in (\mathbb{RP}^n)^\vee : \xi \cap \mathrm{int}(K) = \emptyset\}$ is its polar convex body.

Theorem (F)

If $K \subset L$ are two convex bodies in $\mathbb{RP}^n$, then $\mathrm{vol}_L(K) = \mathrm{vol}_{K^\vee}(L^\vee)$.

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Asymptotics of volume

Asymptotics of volumes of metric balls:

- As $r \to 0$, $\mathrm{vol}(B_K(0,r)) \sim \omega_n^{-1} |K \times K^\circ| r^n$.

- As $r \to \infty$, we have

Theorem (Berck-Bernig-Vernicos, adjusted to Funk metric)

When $K \subset \mathbb{R}^n$ is C^2 and strictly convex, $\mathrm{vol}(B_K(q,r)) \sim c_n \Omega_n(K,q) e^{\frac{n-1}{2} r}$.

Here $\Omega_n(K,q) = \int_{\partial K} \frac{k_x^{1/2}}{\langle x-q, \nu_x\rangle^{(n-1)/2}} d\mathcal{H}^{n-1}(x)$ is the *centro-affine area* of K with center at q.

Remark. Berck-Bernig-Vernicos obtain the result in the Hilbert metric setting under the weaker $C^{1,1}$ assumption and no strict convexity.

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The leading coefficient

A (full) *flag* of P is a chain $f = (\emptyset = f_{-1} \subset f_0 \subset f_1 \subset \cdots \subset f_{n-1} \subset f_n = P)$, where $f_j \in \mathcal{F}_j(P)$ is a j-dimensional face of P.

Theorem (Vernicos-Walsh '18)

In Hilbert geometry, if $P \subset \mathbb{R}^n$ is a convex polytope then

$$\mathrm{vol}_P^H(B_P^H(q,r)) = c_n |\mathrm{Flags}(P)| r^n + o(r^n), \quad r \to \infty.$$

Theorem (F-Vernicos-Walsh)

If $P \subset \mathbb{R}^n$ is a convex polytope, then

$$\mathrm{vol}_P^F(B_P^F(q,r)) = \frac{1}{\omega_n (n!)^2} |\mathrm{Flags}(P)| r^n + o(r^n), \quad r \to \infty$$

The *flag number* $|\mathrm{Flags}(P)|$ of P is a combinatorial analogue of centro-affine surface area.

Theorem (Schütt '91)

If $P \subset \mathbb{R}^n$ is a convex polytope, and P_δ its floating body, then

$$\mathrm{vol}_n(P) - \mathrm{vol}_n(P_\delta) \sim \frac{1}{n! n^{n-1}} |\mathrm{Flags}(P)| \delta \left(\log \frac{1}{\delta} \right)^{n-1}, \quad \delta \to 0^+$$

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$$\mathrm{vol}_n(P) - \mathrm{vol}_n(P_\delta) \sim \frac{1}{n! n^{n-1}} |\mathrm{Flags}(P)| \delta \left(\log \frac{1}{\delta}\right)^{n-1}, \quad \delta \to 0^+$$

The leading coefficient

A (full) *flag* of P is a chain $f = (\emptyset = f_{-1} \subset f_0 \subset f_1 \subset \cdots \subset f_{n-1} \subset f_n = P)$, where $f_j \in \mathcal{F}_j(P)$ is a j-dimensional face of P.

Theorem (Vernicos-Walsh '18)

In Hilbert geometry, if $P \subset \mathbb{R}^n$ is a convex polytope then

$$\mathrm{vol}_P^H(B_P^H(q,r)) = c_n|\mathrm{Flags}(P)|r^n + o(r^n), \quad r \to \infty.$$

Theorem (F-Vernicos-Walsh)

If $P \subset \mathbb{R}^n$ is a convex polytope, then

$$\mathrm{vol}_P^F(B_P^F(q,r)) = \frac{1}{\omega_n (n!)^2}|\mathrm{Flags}(P)|r^n + o(r^n), \quad r \to \infty$$

The *flag number* $|\mathrm{Flags}(P)|$ of P is a combinatorial analogue of centro-affine surface area.

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The Funk-Mahler conjecture

Conjecture (FVW)

For any $0 < r < \infty$, $M_r(K, q) := \omega_n \, vol_K(B_K(q, r))$ is minimized:
- By centered simplices in general.
- By centered Hanner polytopes among centrally-symmetric convex bodies K.

- When $r \to 0$, this becomes Mahler's conjecture.
- When $r \to \infty$ and $K = P$ a polytope, $r^{-n} M_r(P, q) \to c_n |\mathrm{Flags}(P)|$.
- Among all convex polytopes P, $|\mathrm{Flags}(P)|$ is trivially minimized by simplices.

Flag Conjecture (Kalai '89)

For centrally-symmetric P, $|\mathrm{Flags}(P)| \geq 2^n n!$, equality for Hanner polytopes.

Related:

3^d Conjecture (Kalai '89)

Among centrally-symmetric polytopes, the total face number $|\mathcal{F}_0(P)| + \cdots + |\mathcal{F}_n(P)| \geq 3^n$. Equality is attained by Hanner polytopes.

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Lower bounds

The Mahler conjecture is known up to dimension 2 in general (Mahler).
The centrally-symmetric Mahler is known in dimension 3 (Iriyeh-Shibata 2020), for unconditional convex bodies (Saint Raymond '81), zonoids (Reisner '86), some other settings.

Theorem (F-Vernicos-Walsh)

For any $0 < r < \infty$, Hanner polytopes uniquely minimize $\mathrm{vol}_K(B_K(r, 0))$ among all unconditional convex bodies K, .

Taking $r \to \infty$ we make some progress towards Kalai's flag conjecture.

Corollary (F-Vernicos-Walsh)

For unconditional polytopes P, $|\mathrm{Flags}(P)| \geq 2^n n!$.

Equality cases are lost in the limit. But we will say something about equality cases later on.

Theorem (Chambers '22)

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The monodromy group of a polytope

Fix a flag $f \in \text{Flags}(P)$, explicitly

$$\emptyset = f_{-1} = f_0 \subset \cdots \subset f_{n-1} \subset f_n = P.$$

- For $0 \leq i \leq n-1$, there is a unique flag $f' \in \text{Flags}(P)$ such that $f'_j = f_j$ for all $j \neq i$, and $f'_i \neq f_i$.
- The *i-flip* $r_i : \text{Flags}(P) \to \text{Flags}(P)$ is defined by $r_i(f) := f'$. Thus $r_i^2 = \text{id}$.
- The *monodromy group* G_P is generated by all i-flips $r_0, \ldots, r_{n-1}$. It acts on Flags(P).
- Define *the complete flip* $r := r_{n-1} \circ r_{n-2} \circ \cdots \circ r_0 \in G_P$.

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Two terms asymptotics

For a facet $F \in \mathcal{F}_{n-1}(P)$, write $\widehat{F} \in \mathcal{F}_0(P^\circ)$ for the corresponding vertex.

Theorem (F-Vernicos-Walsh)

For a polytope $P \subset \mathbb{R}^n$ with $0 \in \mathrm{int}(P)$ one has

$$\omega_n \operatorname{vol}_P (B_P(R)) = c_0(P)R^n + c_1(P)R^{n-1} + o(R^{n-1}), \quad R \to \infty$$

where

$$c_0(P) = \frac{|\mathrm{Flags}(P)|}{(n!)^2}, \quad c_1(P) = \frac{n}{(n!)^2} \sum_{f \in \mathrm{Flags}(P)} \log\left(1 - \left\langle \widehat{(rf)_{n-1}}, f_0 \right\rangle\right).$$

- If an unconditional polytope has $|\mathrm{Flags}(P)| = |\mathrm{Flags}(H_n)|$, it must have $c_1(P) \geq c_1(H_n)$ (due to known equality cases for finite radius).
- Hanner polytopes maximize $c_1(P)$ among polytopes with $|\mathrm{Flags}(P)| = |\mathrm{Flags}(H_n)|$.

Corollary

If P is unconditional, and $|\mathrm{Flags}(P)| = |\mathrm{Flags}(H_n)| = 2^n n!$, then for every $f \in \mathrm{Flags}(P)$, $-f_0 \in (rf)_{n-1}$.

Does not imply uniqueness of Hanner - any (unconditional) 2-level polytope satisfies this condition.

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The Santaló point

The Santaló point s_K of $K \subset \mathbb{R}^n$ is the unique point $s_K = q \in \mathrm{int}(K)$ such that $|K^q|$ is minimized. One has $s_K = 0$ if and only if 0 is the center of mass of K°.

Theorem (F-Vernicos-Walsh)

- *For each $0 < r < \infty$, there is a unique point $q = s_r(K) \in \mathrm{int}(K)$ that minimizes the Funk volume of $B_K(q, r)$ inside K.*
- *Similarly,*

$$c_1(P,q) = \lim_{R\to\infty} R^{-(n-1)} \left(\omega_n \,\mathrm{vol}_P\left(B_P(q,R)\right) - \frac{|\,\mathrm{Flags}(P)|}{(n!)^2} R^n \right)$$

has a unique minimum at $s_\infty(P) = \lim_{R\to\infty} s_R(P)$.

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Strict convexity of $f(q) := \mathrm{vol}_K(B_K(q,r))$ follows from the strict convexity of $x \mapsto |K^x|$.

Less trivial is showing that f is proper, that is $f(q) \to \infty$ as $q \to \partial K$, without regularity assumptions on K. We use the projective invariance of the Funk volume to squeeze infinitely many disjoint Hilbert balls of fixed radius into a ball centered at ∂K.

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Strict convexity of $f(q) := \operatorname{vol}_K(B_K(q, r))$ follows from the strict convexity of $x \mapsto |K^x|$.

Less trivial is showing that f is proper, that is $f(q) \to \infty$ as $q \to \partial K$, without regularity assumptions on K. We use the projective invariance of the Funk volume to squeeze infinitely many disjoint Hilbert balls of fixed radius into a ball centered at ∂K.

The Santaló point

The Santaló point s_K of $K \subset \mathbb{R}^n$ is the unique point $s_K = q \in \mathrm{int}(K)$ such that $|K^q|$ is minimized. One has $s_K = 0$ if and only if 0 is the center of mass of K°.

Theorem (F-Vernicos-Walsh)

- *For each $0 < r < \infty$, there is a unique point $q = s_r(K) \in \mathrm{int}(K)$ that minimizes the Funk volume of $B_K(q, r)$ inside K.*
- *Similarly,*

$$c_1(P, q) = \lim_{R\to\infty} R^{-(n-1)} \left(\omega_n \mathrm{vol}_P \left(B_P(q, R) \right) - \frac{|\mathrm{Flags}(P)|}{(n!)^2} R^n \right)$$

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Upper bound

Conjecture (Funk-Blaschke-Santaló)

Given $0 < r < \infty$, $\min_{q \in K} \mathrm{vol}_K(B_K(q,r))$ is uniquely maximized by ellipsoids.

Motivation: • For $r \to 0$ it is the Blaschke-Santaló inequality.
• For $r \to \infty$, it is the centro-affine isoperimetric inequality of Lutwak: $\Omega_n(K, c.m.) \leq \Omega_n(B^n)$.

Theorem (Berck-Bernig-Vernicos '10, Vernicos-Yang '19)

For a $C^{1,1}$ convex body K and $0 \in \mathrm{int}(K)$, the Hilbert ball $B_K^H(R,0)$ has volume

$$\mathrm{vol}_K^H(B_K^H(R,0)) \sim \frac{1}{n-1} \mathcal{C}_0(K) e^{(n-1)R}, \quad R \to \infty.$$

The centro-projective surface area $\mathcal{C}_0(K)$ ia uniquely maximized by ellipsoids.

The Colbois-Verovic volume entropy conjecture:.

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In Hilbert geometry, $\limsup_{r \to \infty} \frac{\log \mathrm{vol}_K^H(B_K^H(q,r))}{r} \leq n-1.$

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Theorem (F, jdg '22+)

Among unconditional convex bodies K, ellipsoids uniquely maximize $\mathrm{vol}_K(B_K(r,0))$ for any $0 < r < \infty$.

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Among m-polygons $P \subset \mathbb{R}^2$, affine images of the regular m-polygon uniquely maximize $c_1(P, s_\infty(P))$.

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Functional inequalities

Functional Funk-Blaschke-Santaló conjecture

For even $\phi : \mathbb{R}^n \to \mathbb{R}$, and $0 < \lambda < 1$ one has

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} e^{-\phi(x) - \mathcal{L}\phi(\xi) + \lambda \langle x, \xi \rangle} dx d\xi \leq \frac{(2\pi)^n}{(1-\lambda^2)^{n/2}}$$

with equality only for $e^{-\phi}$ gaussian.

- Proved in [F, jdg '22+] for unconditional ϕ.

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For convex even $\phi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, and $0 < \lambda < 1$ one has

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Wild speculations

Yet another conjecture of Kalai

A centrally-symmetric polytope P satisfies $|\mathrm{Flags}(P)| \geq \frac{n!^2}{2^n}|P||P^\circ|$.

Aiming for a finite radius version, we may boldly propose

A reverse Bishop-Gromov-type conjecture (FVW)

Let $0 < r < R < \infty$, and $K \subset \mathbb{R}^n$ a centrally-symmetric convex body. Then

$$\frac{\mathrm{vol}_K(B_K(R))}{\mathrm{vol}_K(B_K(r))} \geq \frac{\mathrm{vol}_{H_n}(B_{H_n}(R))}{\mathrm{vol}_{H_n}(B_{H_n}(r))}.$$

Kalai's conjecture follows when $r \to 0, R \to \infty$.

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Direct Bishop-Gromov is false

Bishop-Gromov theorem

Let $0 < r < R < \infty$, M complete Riemannian with $\mathrm{Ric}_M \geq (n-1)K$. Let M_K be the model space of that curvature. Then

$$\frac{\mathrm{vol}_M(B_M(p,R))}{\mathrm{vol}_M(B_M(p,r))} \leq \frac{\mathrm{vol}_{M_K}(B_{M_K}(p_M,R))}{\mathrm{vol}_{M_K}(B_{M_K}(p_M,r))}.$$

The direct analogue of Bishop-Gromov is provably false:

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Let $0 < r < R < \infty$, $K \subset \mathbb{R}^n$ a centrally-symmetric convex body, E_n an ellipsoid. Then

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This is false: Taking $R, r \to 0$, this implies the inequality

$$\int_{K\times K^\circ} \langle x,\xi\rangle^2 dx d\xi \leq \frac{n}{(n+2)^2}|K||K^\circ|,$$

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The End

Thanks for listening!

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Can you prove or disprove it?

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