The L_p-Minkowski problem - Old and New results

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Cortona, June 30, 2023

Founding Fathers

- MINKOWSKI
- Firey
- Lutwak

Minkowski combination, Brunn-Minkowski inequality

$K, C \subset \mathbb{R}^n$ convex bodies, $\alpha, \beta > 0 \Longrightarrow h_{\alpha K + \beta C} = \alpha h_K + \beta h_C$

$$\begin{aligned} \alpha K + \beta C &= \{\alpha x + \beta y : x \in K,\ y \in C\} \\ &= \{x \in \mathbb{R}^n : \langle u, x \rangle \leq \alpha h_K(u) + \beta h_C(u)\ \forall u \in S^{n-1}\} \end{aligned}$$

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Brunn-Minkowski inequality $\alpha, \beta > 0$

$$V(\alpha\, K + \beta\, C)^{\frac{1}{n}} \geq \alpha\, V(K)^{\frac{1}{n}} + \beta\, V(C)^{\frac{1}{n}}$$

with equality iff K and C are homothetic ($K = \gamma C + x$, $\gamma > 0$).

Equivalent form $\lambda \in (0, 1)$

$$V((1 - \lambda)\, K + \lambda\, C) \geq V(K)^{1-\lambda} V(C)^{\lambda}.$$

Surface area measure = First variation of volume

S_K - surface area measure on S^{n-1} of a convex body $K \subset \mathbb{R}^n$

$$\lim_{\varrho \to 0^+} \frac{V(K + \varrho\, C) - V(K)}{\varrho} = \int_{S^{n-1}} h_C \, dS_K$$

for any convex body $C \subset \mathbb{R}^n$ (e.g. $V(K) = \frac{1}{n} \int_{S^{n-1}} h_K \, dS_K$)

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Minkowski inequality (equivalent to Brunn-Minkowski)
$K, C \subset \mathbb{R}^n$ convex body with $V(K) = V(C) \Longrightarrow$

$$\int_{S^{n-1}} h_C \, dS_K \geq \int_{S^{n-1}} h_K \, dS_K \quad (= nV(K)\,)$$

with equality $\Longleftrightarrow$ $C = K + x$

- ∂K is C^2_+, $f_K(\nu_K(x)) = \kappa_{\partial K}(x)^{-1}$ for $x \in \partial K \Longrightarrow$

$$dS_K = f_K \, d\mathcal{H}^{n-1}$$

- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i).$$

Minkowski problem

μ Borel measure on S^{n-1}

$\exists K \subset \mathbb{R}^n$ convex body with $\mu = S_K \iff$

- $\mu(L \cap S^{n-1}) < \mu(S^{n-1})$ for any linear $(n-1)$-subspace $L \subset \mathbb{R}^n$
- $\displaystyle\int_{S^{n-1}} u \, d\mu(u) = o$

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Monge-Ampere equation on S^{n-1}:

$$\det(\nabla^2 h + h \, I_{n-1}) = f$$

Regularity Nirenberg, Pogorelov: $f > 0$ C^k, $k \geq 2 \Longrightarrow h$ is C^{k+2}

Caffarelli: $f > 0$ C^α for $\alpha \in (0,1) \Longrightarrow h$ is $C^{2,\alpha}$

To solve the Minkowski problem ,

- Minimize $\int_{S^{n-1}} h_C \, d\mu$ under the condition $V(C) = 1$
- Uniqueness up to translation comes from uniqueness in the Minkowski inequality

Logarithmic Minkowski problem - Cone volume measure

$dV_K = \frac{1}{n} h_K dS_K$ - cone volume measure on S^{n-1} if $o \in K$
(Firey, 1974) - L_0 surface area measure

- K polytope, $F_1, \ldots, F_k$ facets, u_i exterior unit normal at F_i

$$V_K(\{u_i\}) = \frac{h_K(u_i)\mathcal{H}^{n-1}(F_i)}{n} = V(\mathrm{conv}\{o, F_i\}).$$

- $V_K(S^{n-1}) = V(K)$.

Monge-Ampere type differential equation on S^{n-1} for $h = h_K$ if μ has a density function f (Firey, 1974):

$$h \det(\nabla^2 h + h\, I_{n-1}) = f$$

Naor, Werner, Paouris, Stancu... used it for example for L_p balls

Firey's Gauss curvature flow

Firey's worn stone (1974), $\alpha > 0$

$$\frac{\partial}{\partial t} X(x,t) = -\kappa(x,t)^{\alpha} \nu(x,t)$$

Actually, Firey's original problem when $\alpha = 1$

Andrews, Guan, Ni (2016) $\alpha > \frac{1}{n+1} \implies$ Normalized flow converges to a self similar soliton satisfying

$$h^{\frac{1}{\alpha}} \det(\nabla^2 h + h\, I_{n-1}) = \mathrm{constant}.$$

Brendle, Choi, Daskalopoulos (2017) Unique solution is the ball

$\alpha = \frac{1}{n+1} \implies$ centered ellipsoids solutions (Calabi, Andrews)
$h^{n+1} \det(\nabla^2 h + h\, I_{n-1}) =$centro-affine curvature function invariant under $\mathrm{SL}(n)$

Lutwak's L_p surface area measures

L_p surface area measures (Lutwak 1990) $p \in \mathbb{R}$

$$dS_{K,p} = h_K^{1-p}\, dS_K = n h_K^{-p}\, dV_K$$

Examples

- $S_{K,1} = S_K$
- $S_{K,0} = nV_K$
- $S_{K,-n}$ related to the $\mathrm{SL}(n)$ invariant curvature $\frac{\kappa_K(u)}{h_K(u)^{n+1}}$

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Remark Possibly $o \in \partial K$ where $dS_{K,p} = f d\mathcal{H}^{n-1}$ and $f > 0$ and f is C^α if $-n+2 < p < n$

Variational approach

- Minimize "$\int_{S^{n-1}} h_C^p\, d\mu$" under the condition $V(C) = 1$
- Weak approximation by "nice" measures

Lutwak's L_p Minkowski problem $\sim$ 1990

L_p Minkowski problem $h_K^{1-p} dS_K = d\mu$, $o \in K$

Monge-Ampere on S^{n-1} for $h = h_K$ if μ has a density function f:

$$h^{1-p} \det(\nabla^2 h + h\, I_{n-1}) = f$$

- $p = 1 \Longrightarrow$ Minkowski problem
- $p = 0 \Longrightarrow$ Logarithmic Minkowski problem
- $p = -n \Longrightarrow$ Determining Centro-affine curvature

State of art

- $p > 1$, $p \neq n$: Guan&Li, Chou&Wang, Hug&LYZ
- $0 < p < 1$: Chen&Li&Zhu "almost complete"
- $p = 0$: positive results by Chen&Li&Zhu
- $-n < p < 0$: $f \in L_{\frac{n}{n+p}}(S^{n-1})$ (Bianchi&B&Colesanti)
- $p < -n$: $f > 0$ C^3 by Guang, Li, Wang

When the support is lower dimensional

$K \subset \mathbb{R}^n$ convex body, $o \in K$
$L \subset \mathbb{R}^n$ linear subspace, $2 \leq d+1 = \dim L \leq n-1$

- $p < 1 \implies S_{p,K}|_L \neq \mathcal{H}^d$ (Saroglou)
- $0 < p < 1$, $\operatorname{lin}\operatorname{supp}\mu = L \cap S^{n-1}$ and $\exists v \in L \cap S^{n-1}$ s.t. $\langle u, v\rangle \leq 0$ for $u \in \operatorname{supp}\mu \implies \mu = S_{p,K}$ (Bianchi, Colesanti, B., Yang)

Open problem Characterize $S_{p,K}$ if $p \in (0,1)$ & $\operatorname{supp}\mu \subset L \cap S^{n-1}$

When f is almost constant

$$h^{1-p}\det(\nabla^2 h + h\, I_{n-1}) = f, \; -n < p < 1$$

- Uniqueness holds if f =constant (Brendle&Choi&Daskalopoulos (2017), Saroglou (2022) with Steiner symmetrization, Milman&Ivaki (2023) elegant argument)
- Stability (Ivaki, 2022): f is close to be a constant + extra conditions $\Longrightarrow$ solution close to be a ball
- Uniqueness near constants (B., Saroglou 2023+) f is $C^{0,a}$ close to be a constant and $p \in [0, 1)$ $\Longrightarrow$ solution is unique ($p = 0$ and $n = 3$ due to Chen&Feng&Liu)

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Open problem $\lambda^{-1} \leq h^{1-p}\det(\nabla^2 h + h\, I_{n-1}) \leq \lambda$ and $p \in [0,1)$ $\Longrightarrow^{?}$ $h \leq C(\lambda, n, p)$

Known $p \in [0,1)$ and $n = 3$ (Chen&Feng&Liu and B&Saroglou)
$p = 0$ and $n = 4$ (B&Saroglou)

Uniqueness in the L_p-Minkowski problem

L_p-Minkowski problem on S^{n-1}

$$h^{1-p}\det(\nabla^2 h + h\, I_{n-1}) = f \geq 0$$

- Uniqueness holds if $p > 1$ (Chou&Wang, Hug&Lutwak&Yang&Zhang)
- No uniqueness in general if $p < 1$ (Chen&Li&Zhu)

Even L_p-Minkowski problem on S^{n-1}

- Uniqueness Conjectured if $0 < p < 1$
- Uniqueness Conjectured if $p = 0$ and $f > 0$ is C^∞
- No uniqueness if $p < 0$ (Li&Liu&Lu, E. Milman)
- Uniqueness if $p_n < p < 1$ and $f > 0$ is C^∞ where $p_n = 1 - \frac{c}{n \log n} \in (0,1)$ (Chen&Huang&Li&Liu, Kolesnikov&Milman, Putterman)

Even cone volume measures

Theorem (B, Lutwak, Yang, Zhang, 2013)

Let μ be an even Borel measure on S^{n-1}.
$\mu = V_K$ for some o-symmetric convex body K iff

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \mu(S^{n-1})$ *for any* $L \neq \{o\}, \mathbb{R}^n$

(ii) *If equality holds for some L, then* $\operatorname{supp} \mu \subset L \cup L'$ *for some complementary L'*

Idea $\mu = \alpha V_{\widetilde{C}}$ if $\widetilde{C}$ minimizes $\int_{S^{n-1}} \log h_C \, d\mu$ assuming $V(C) = 1$

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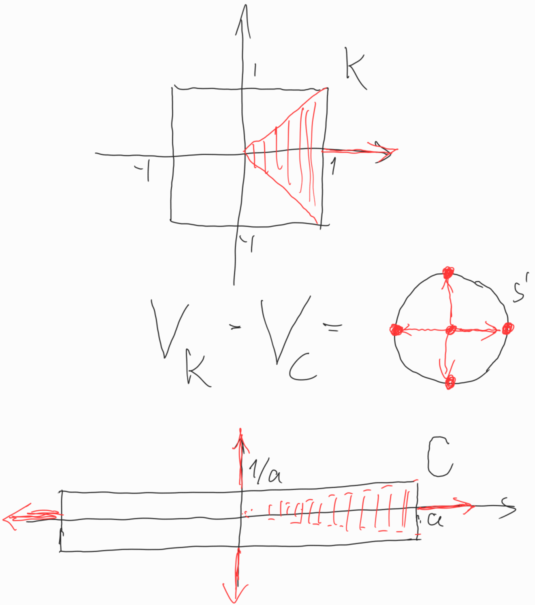
Conjecture (Uniqueness)

$V_K = V_C$ for o-symmetric convex bodies K and C with $V(K) = V(C)$ iff K and C have dilated direct summands; namely, $K = K_1 \oplus \ldots \oplus K_m$ and $C = C_1 \oplus \ldots \oplus C_m$ with $K_i = \lambda_i C_i$ for $\lambda_1, \ldots, \lambda_m > 0$.

Coinciding cone volumes



Logarithmic (L_0) Minkowski conjecture

Conjecture (B, Lutwak, Yang, Zhang)

$K, C \subset \mathbb{R}^n$ o-symmetric convex bodies, $V(K) = V(C) \Longrightarrow$

$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K. \qquad (1)$$

Assuming K is smooth, equality holds $\Longleftrightarrow K = C$.

Remark For even C_+^∞ data, uniqueness of the solution of the Log-Minkowski problem $\Longleftrightarrow$ equality holds in (1) only if $K = C$.

Known results

- $n = 2$
- K is close to some ellipsoid (Colesanti&Livshyts&Marsiglietti, Kolesnikov&Milman, Chen&Huang&Li&Liu)
- K, C have complex symmetry (Rotem)
- K, C - hyperplane symmetry (Saroglou, B&Kalantzopoulos)
- K zonoid (van Handel)

L_p Minkowski inequality/conjecture

L_p Minkowski inequality/conjecture
inequality if $p \geq 1$ by Minkowski, Firey (1962)
conjecture if $0 < p < 1$ and K, C o-symmetric

$$\int_{S^{n-1}} \left(\frac{h_C}{h_K}\right)^p dV_K \geq V(K)\left(\frac{V(C)}{V(K)}\right)^{\frac{p}{n}}$$

with equality if $p \neq 1 \iff K$ and C are dilates.

- Proved if $p_n \leq p < 1$ where $p_n = 1 - \frac{c}{n \log n}$ (Chen&Huang&Li&Liu, Kolesnikov&Milman, Putterman)
- if inequality holds for some $p \geq 0$, then holds for $q > p$
- if $0 < p < 1$, inequality for o-symmetric convex bodies equivalent to unique even solution of L_p-Minkowski problem (enough to consider C_+^∞ case)

Logarithmic (L_0) Brunn-Minkowski conjecture

$\lambda \in [0,1]$, $o \in K, C$

$$(1-\lambda)K +_0 \lambda C = \{x \in \mathbb{R}^n : \langle u, x\rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda} \ \forall u \in S^{n-1}\}$$

$$\lambda K +_0 (1-\lambda)C \ \subset \ \lambda K + (1-\lambda)C$$

Conjecture (Logarithmic Brunn-Minkowski conjecture)

$\lambda \in (0,1)$, K, C o-symmetric

$$V((1-\lambda)K +_0 \lambda C) \geq V(K)^{1-\lambda} V(C)^{\lambda}$$

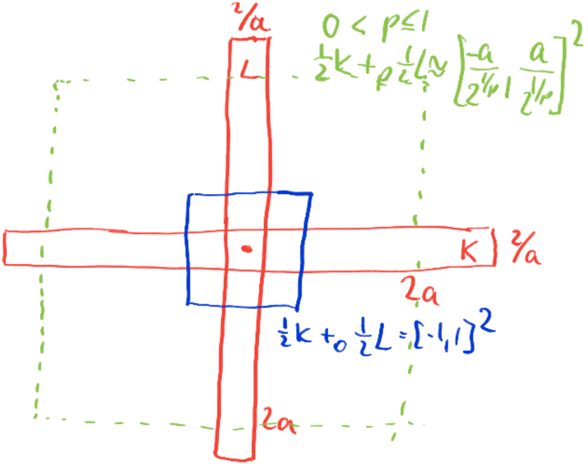
with equality iff K and C have dilated direct summands.

$$4^{-n}|K|^{1-\lambda}|C|^{\lambda} \leq |(1-\lambda)K +_0 \lambda C| \leq n^{2n}|K|^{1-\lambda}|C|^{\lambda}$$

Known results

- $n = 2$
- K, C are close to a fixed ellipsoid (Kolesnikov&Milman, Colesanti&Livshyts&Marsiglietti, Chen&Huang&Li&Liu)
- K, C have complex symmetry (Rotem)
- K, C - hyperplane symmetry (Saroglou, B&Kalantzopoulos)

Lp sum of coordinate boxes



$$\tfrac{1}{2}K +_p \tfrac{1}{2}L = \{x \in \mathbb{R}^2 : \langle x,u\rangle \le \left(\tfrac{1}{2}h_K(u)^p + \tfrac{1}{2}h_L(u)^p\right)^{\frac{1}{p}} \ \forall u \in S^1\} \quad p>0$$

$$\tfrac{1}{2}K +_0 \tfrac{1}{2}L = \{x \in \mathbb{R}^2 : \langle x,u\rangle \le \sqrt{h_K(u)\, h_L(u)} \ \forall u \in S^1\}$$

L_p Brunn-Minkowski inequality/conjecture

$p > 0$, $\lambda \in (0,1)$, $o \in \mathrm{int}K, \mathrm{int}L$

$\lambda K +_p (1-\lambda)L = \{x \in \mathbb{R}^n : \langle u, x\rangle^p \leq \lambda h_K(u)^p + (1-\lambda)h_L(u)^p \ \forall u\}$

$p \geq 1$ $\ h_{\lambda K +_p (1-\lambda)L} = \left(\lambda h_K^p + (1-\lambda)h_L^p\right)^{1/p}$

L_p BM inequality($p \geq 1$)/conjecture($0 < p < 1$,o-symm)

$$V((1-\lambda)K +_p \lambda L)^{\frac{p}{n}} \geq (1-\lambda)V(K)^{\frac{p}{n}} + \lambda V(L)^{\frac{p}{n}}$$

with equality iff K and L are dilated. Equivalent

$$V(\lambda K +_p (1-\lambda)L) \geq V(K)^{\lambda} V(L)^{1-\lambda}$$

Known if $p > 1$ (Firey, 1962), $p = 1$ (Minkowski),
$p_n < p < 1$ where $p_n = 1 - \frac{c}{n \log n} \in (0,1)$ (Chen&Huang&Li&Liu, Kolesnikov&Milman, Putterman)