Intrinsic volumes of Kähler manifolds

(joint with J. Fu, G. Solanes and T. Wannerer)

Andreas Bernig

Goethe-Universität Frankfurt am Main

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Hadwiger's theorem

- Steiner's formula:

$$\mathrm{vol}(K+tB)=\sum_{k=0}^{n}\mu_k(K)\omega_{n-k}t^{n-k}.$$

- A valuation on a vector space V satisfies
$\mu(K\cup L)+\mu(K\cap L)=\mu(K)+\mu(L)$.
- Example: the intrinsic volume μ_k is a valuation.
- μ_k is invariant under Euclidean motions.
- For fixed A, the valuation $K\mapsto \mathrm{vol}(K+A)$ is a translation invariant valuation.
- $\mathrm{Val}:=\{$continuous, translation invariant valuations$\}$

Hadwiger's theorem

The space of $\mathrm{Val}^{\mathrm{O}(n)}$ of continuous, Euclidean motion invariant scalar-valued valuations has the intrinsic volumes as basis.

Product structure

Alesker

On some dense subspace Val^∞ of smooth valuations on V, there is a product structure characterized by the following property: If $\phi_i(K) = \mathrm{vol}_n(K + A_i)$ with A_i compact, convex, smooth with positive curvature, then

$$\phi_1 \cdot \phi_2(K) = \mathrm{vol}_{2n}(\Delta K + A_1 \times A_2),$$

where $\Delta : V \to V \times V$ is the diagonal embedding.

Example: $\mu_k \cdot \mu_l = \begin{bmatrix} k+l \\ k \end{bmatrix} \mu_{k+l}$. $\mathrm{Val}^{\mathrm{O}(n)} \cong \mathbb{R}[t]/(t^{n+1})$.

Kinematic formulas

Kinematic formulas

Let G be a subgroup of $\mathrm{O}(n)$ that acts transitively on the unit sphere. Then the space Val^G of continuous, translation invariant and G-invariant valuations is finite-dimensional. If $\phi_1, \ldots, \phi_N$ is a basis, there are kinematic formulas

$$\int_{\bar{G}} \phi_i(K \cap \bar{g}L) d\bar{g} = \sum_{k,l} c^i_{k,l} \phi_k(K) \phi_l(L).$$

B'-Fu 2006

The kinematic formulas determine a coalgebra structure on Val^G. The associated algebra structure on the dual space Val^{G*} is naturally isomorphic to the algebra $(\mathrm{Val}^G, \cdot)$.

In short: the knowledge of the kinematic formulas is equivalent to the knowledge of the algebra structure on Val^G.

Weyl's principle

Weyl's principle

Let $M \subset \mathbb{R}^N$ be a compact n-dimensional submanifold. Then the volume of the r-tube is given for small enough $r > 0$ by

$$\mathrm{vol} M_r = \sum_{k=0}^{n} \mu_k(M) \omega_{N-k} r^{N-k}.$$

The coefficients $\mu_k(M)$ are the intrinsic volumes of M. They only depend on the inner metric and not on the embedding! They can be defined as valuations on M, denoted by μ_k^M. If $(M, g) \hookrightarrow (\tilde{M}, \tilde{g})$ is an isometric embedding, then

$$\mu_k^{\tilde{M}, \tilde{g}}|_M = \mu_k^{M,g}.$$

Example: $\mu_0^M = \chi$ is the Euler characteristic. μ_n^M the volume.
$\mu_{n-2}(M) = \frac{1}{4\pi} \int s d\mathrm{vol}$ total scalar curvature

Valuation on manifolds

- A valuation on a manifold X is a finitely additive functional $\mu : \mathcal{P}(X) \to \mathbb{R}$, where $\mathcal{P}(X)$ is the set of differentiable polyhedra on X (alternatively, manifolds with corners).
- Smooth valuations are defined in terms of differential forms on the cosphere bundle.
- Chern: the Euler characteristic χ is a smooth valuation.
- There is a product structure on the space of smooth valuations (Alesker-Fu 2008, Alesker-Bernig 2012). The Euler characteristic is the neutral element.

Isotropic spaces

- M Riemannian manifold, G a subgroup of the isometry group. (M, G) is called isotropic if G acts transitively on the unit tangent bundle.
- Examples:
$(\mathbb{R}^n, \overline{\mathrm{SO}(n)}), (S^n, \mathrm{SO}(n+1)), (\mathbb{C}^n, \overline{\mathrm{U}(n)}), (\mathbb{CP}^n, \mathrm{Isom}), (\mathbb{H}P^n, \mathrm{Isom})$.
- The space of G-invariant smooth valuations is finite-dimensional. There are kinematic formulas:

$$\int_G \phi_i(A \cap gB)dg = \sum_{k,l} c_{k,l}^i \phi_k(A)\phi_l(B)$$

The structure coefficients are encoded by the product structure as in the flat case.
- Example: $\mathcal{V}(S^n)^{\mathrm{SO}(n+1)} \cong \mathbb{R}[t]/(t^{n+1}) \cong \mathrm{Val}^{\mathrm{O}(n)}$

Hermitian case

- Hermitian case $(\mathbb{C}^n, \overline{\mathrm{U}(n)})$: Park 2002, Alesker 2003, Fu 2006, B'-Fu 2012, Wannerer 2014...
- Alesker: $\dim \mathrm{Val}^{\mathrm{U}(n)} = \binom{n+2}{2}$
- Fu:
$$\mathrm{Val}^{\mathrm{U}(n)} \cong \mathbb{R}[t,s]/(f_{n+1}, f_{n+2}),$$
with
$$\log(1+tx+sx^2) = \sum_{i=1}^{\infty} f_i(t,s)x^i.$$
- B'-Fu 2011: explicit kinematic formulas

Complex projective space

Complex projective space $\mathbb{CP}^n$ (complex lines in $\mathbb{C}^{n+1}$): Park 2002, Abardia-Gallego-Solanes 2012, B'-Fu-Solanes 2014,...

B'-Fu-Solanes 2014

$$\mathcal{V}(\mathbb{CP}^n)^{\mathrm{Isom}} \cong \mathrm{Val}^{\mathrm{U}(n)}.$$

- Comes out of the computation, without a satisfactory explanation.
- Can be translated into explicit kinematic formulas.

More general viewpoint: Kähler manifolds

- A Kähler manifold is a Riemannian manifold (M, g) with an integrable almost-complex structure J such that the fundamental form $F = g(J\bullet, \bullet)$ is a closed 2-form.
- Combines Riemannian, symplectic and complex geometry.
- Examples: $\mathbb{C}^n, \mathbb{CP}^n$, Grassmann manifolds, flag manifolds, (smooth) projective varieties

Weyl principle for Kähler manifolds

B'-Fu-Solanes-Wannerer 2021

For any Kähler manifold M of complex dimension n, there is a canonical subalgebra $\mathrm{KLK}(M) \subset \mathcal{V}(M)$, the Kähler-Lipschitz-Killing algebra, isomorphic to $\mathrm{Val}^{\mathrm{U}(n)}$, with the property that if $M' \hookrightarrow M$ is a Kähler embedding then the natural restriction map $\mathcal{V}(M) \to \mathcal{V}(M')$ restricts to a natural surjection $\mathrm{KLK}(M) \to \mathrm{KLK}(M')$.

Some problems and solutions

Two possible approaches:

- In the Riemannian case, Fu and Wannerer have given a complete description of invariant differential forms that can be built in Cartan's calculus. In the Kähler case, this seems hopeless, as the dimension of this space is really large, even for manifolds of small dimension.
- If M is isometrically and holomorphically embedded into some $\mathbb{C}^{n+N}$, we can restrict invariant valuations from $\mathbb{C}^{n+N}$. In contrast to the Riemannian case, there is no Nash-type theorem for Kähler manifolds, i.e. not every Kähler manifold can be embedded in flat space.

Both approaches fail, but a mixture of both approaches works well.

Algebraic Kähler curvature tensors I

M Riemannian manifold. Then the curvature tensor R_x satisfies:

1. $R(Y, X, Z, W) = -R(X, Y, Z, W)$
2. $R(Z, W, X, Y) = R(X, Y, Z, W)$
3. $R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$ (Bianchi identity)
4. If M is Kähler: $R(JX, JY, Z, W) = R(X, Y, Z, W)$

If R is a $(0,4)$-tensor on $\mathbb{C}^n$ satisfying these properties, we call it an algebraic Kähler tensor. If there exists some n-dimensional complex submanifold of some $\mathbb{C}^{n+N}$ whose curvature tensor at 0 is the given tensor R, we call R embedded.

Algebraic Kähler tensors II

Put $W := \mathrm{Sym}^2_{\mathbb{C}}(\mathbb{C}^n)^*$ and define a map $\theta : \mathrm{Sym}^2_{\mathbb{R}} W \to \otimes^4(\mathbb{C}^n)^*$ by

$$\begin{aligned}\theta(k \cdot l)(X, Y, Z, W) \\ &= \frac{1}{2}\mathrm{Re}[k(X,Z)\overline{l(Y,W)} + l(X,Z)\overline{k(Y,W)} \\ &\qquad - k(X,W)\overline{l(Y,Z)} - l(X,W)\overline{k(Y,Z)}]\end{aligned}$$

Pointwise embedding lemma

The set of embedded algebraic Kähler curvature tensors equals the sums of squares cone, in particular it is a full-dimensional convex cone.

Any formula which depends polynomially on the curvature tensor and which holds true for complex submanifolds of $\mathbb{C}^{n+N}$ will also hold on arbitrary n-dimensional Kähler manifolds.

Construction of the Kähler intrinsic volumes

- Use double forms.
- The metric tensor is a double form of type $(1, 1)$. The Riemann curvature tensor is a double form of type $(2, 2)$. Using the complex structure, there is another double form of type $(2, 2)$, which is modelled on the curvature tensor of the complex projective space.
- The Levi-Civita connection induces a double form of type $(1, 1)$ on the sphere bundle, called the connection form.
- Take convenient products and linear combinations of these building blocks.
- Use combinatorial identities involving determinants of Catalan numbers.

Complex projective space

New insights from the Kähler approach even in the case of complex projective space:

- There is a canonical isomorphism

$$\mathcal{V}(\mathbb{CP}^n)^{\mathrm{Isom}} \cong \mathrm{Val}^{\mathrm{U}(n)}.$$

- Global kinematic formulas, semilocal kinematic formulas and local kinematic formulas on Hermitian space $\mathbb{C}^n$, on complex projective space $\mathbb{CP}^n$, and on complex hyperbolic space $\mathbb{CH}^n$ are formally identical.
- Is the same true in the quaternionic setting?